Project-Based Mathematical Investigation For Prospective K-8 Teachers: Students Produce Original Approaches to the Generation of Pythagorean Triples

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Introduction

The setting of this research was an initial pilot version of a reformed mathematics class for pre-service K-8 teachers at the University of Texas at El Paso (UTEP). This course, entitled "Properties of Real Numbers," is the last mathematics content course that is required for all students certifying as K-8 teachers regardless of specialization. Traditionally this course focused on running through the algebraic structures of the integers, rationals and real number systems in a largely formal way with some attention to problem solving activities. The one prerequisite course is a standard college course that moves quickly through traditional algebra (linear and quadratic), logarithms, and some probability, where pre-service teachers are mixed with students majoring in business and nursing.

In an effort to rethink possible mathematical options for the these future teachers, an initial research experiment was conducted as part of the first phase of a National Science Foundation project known in El Paso as the Partnership for Excellence in Teacher Education (PETE), which is one of a number of NSF's preservice teacher collaborative around the nation. This experiment was conducted with one section of 36 students in the Fall of 1995. The Department of Mathematical Sciences at UTEP and the College Education agreed to allow for complete freedom during this pilot to experiment with new curriculum and forms of assessment.

El Paso is an industrial urban center in far west Texas situated on the Mexican border next to Ciuadad Jaurez, Mexico, the fourth largest city in Mexico. El Paso is one the poorest cities in the U.S.A. and is approximately 70% Mexican-American. It is isolated by many miles of barren desert in all directions. 85% of the students at UTEP are drawn from the city of El Paso and 80% of the teachers in the El Paso area public schools are graduates of UTEP. The situation has been described as a "closed loop." In the pilot class to be discussed, 25 of the 36 students were Mexican-Americans.

The Nature the Classroom Learning Environment

The learning environment of this experimental mathematics class was modeled largely on the theories of Jere Confrey which are based largely on a radical constructivist framework with elements of social Vygotskian theory (1994b). Several pieces of the curriculum were taken directly from Confrey's work on student cognition of ratio and proportion (1994a). One main goal was to create what Confrey calls a "balanced dialogue between grounded activity and systematic inquiry." The other main goal was to bring student voice and perspective to the foreground. These intentions were implemented through a series of weekly projects where the students engaged with challenging mathematical investigations that all stemmed from direct physical situations.

Assessment in the class was based on written project reports that were turned in weekly. These reports were returned with written comments and students then had the option of rewriting each report for a higher grade. Other than the 15 week semester itself, there was no limit on the number of times that a report could be rewritten. A student's grade was based on top grades eventually achieved. Project reports were given one of three grades.

√- Some engagement with the problem, but substantial questions remain.

√ A well reasoned explanation, but some questions remain.

√+ A complete and thorough explanation, no further questions.

The three possible project grades are a modified version of the situation faced by mathematical researchers where progress usually begins with some special cases (√-), then moves to an explanation of all but a few special cases (√), and ends with a through argument (√+). The student's in this class were not held to the linguistic standards of formal mathematical proof, but rather to a standard of peer review by their classmates in the context of the questions and comments of the instructor. A √+ was given when proof beyond a reasonable doubt was achieved in the social context of the class. In most cases such arguments could be easily transformed into formal proofs by a professional mathematician operating in his/her own social setting. Students who eventually achieved a √ on 80% of the projects got a grade of C or better. Students who eventually achieved a √+ on 90% of the projects received an A. Students were informed of this policy at the outset of the class.

The mathematical content of the curriculum fell into three basic sections. The first several projects contained a variety of situations where integer data was collected from counting situations and the students sought ways to make predictions (e.g. the tower of Hanoi). Both recursive and explicit statements were acceptable as long as they were well connected to the situation. Some study of the relations between recursion and explicit statements were discussed using student generated strategies. The second set of projects focused on the relations between integers, ratios, and geometry. Beginning with geoboard problems (i.e. lattice geometry) students investigated lengths, areas, scaling, similarity, and quadratic irrational ratios. For example, they were asked to find a strategy for finding all possible rescalings of a figure on a geoboard of arbitrary size. The final projects dealt with data and statistical distributions from games and the algebraic structure of symmetry transformations. For example, the symmetry group of a square was generated from the question, "how many ways can you pick up a cardboard square and return it to the same place on the table?"

Class time was spent in the following ways: short introductions to new projects, group explorations, class presentations after the first round of written reports, and brief lectures on the historical and cultural background of concepts. Symbolic notation was introduced only as needed to deal with concepts that arose in the group explorations. Student presentations were used to give struggling students a chance to learn from their classmates and to push the presenters to further articulate their ideas and arguments, thus allowing everyone to improve on their rewritten reports.

Student voice and perspective became the focus of most of the class discussions as various strategies and predictions were tested in a wider problem solving context and in relation to the social setting of the students. The content of the curriculum was controlled by the instructor (myself) only to the extent that I provided the tools and asked the initial questions. The ultimate direction of further investigations was largely shaped by the questions that arose in the class. Many of the initial projects were adapted directly from my studies of the history of mathematics although only minimal historical background was provided directly to the students (Dennis, in press). Students who desired more cultural and historical background were encouraged to engage in outside research or to take a newly-restructured course in the history of mathematics.

The Pythagorean Project Emerges

In order to see how this process worked, this paper will examine in detail how a particular set of questions and strategies developed in the class. In particular, the work of two students on one project will be described in detail. The focus project itself was not one of the curricular intentions at the outset, but was developed in the middle of the class in response to questions that arose on the first set of geoboard (lattice geometry) explorations. Initially students were asked to find all possible different sized squares that can be made with one rubber band on a geoboard[1] along with their areas. After constructing squares of area 1, 4, 9, and 16 square units they found others of area 2, 5, 8, and 10 square units, that sit at odd angles with respect to the lattice. Eventually most students showed that any line between any two points on a lattice can be used as a side of a square with all four corners occuring as lattice points, although articulating an argument for why this was true was, at first, quite difficult for them.

They were then asked to find the lengths of the sides of these squares using the distance unit of the lattice. Most students attempted to measure the sides by making a ruler in the appropriate lattice units. Very few students used any form of square root calculation in the form of either the distance formula or the Pythagorean theorem, although this had been taught to all of them in previous formal mathematics classes. One student was angry that I would even indirectly imply that he should have done this. He said, "how can you expect us to use such an idea here when all we have ever done is memorize formulas to pass tests; the minute the test is over, we brain dump that stuff. We've never used any of it except on math tests."

In response to the situation we discussed various forms of measurement and estimation. I showed them an ancient Hindu method for finding integer fractions that approach the value of any square root. Although difficult for them at first, this method has strong ties to basic geometrical ideas in that it produces a series of trimmings which construct a square of any given area (integer areas in our case). They all mastered this technique and many were intrigued that they could now find a fractional approximation to a square root that was far more accurate than the 9 place decimal given by the square root button on their calculators.

As we continued our investigations of scalings and ratios on the lattice using graph paper as an enlarged geoboard, some students noticed that not all diagonal lengths required a tedious square root calculation. One student said, "hey sometimes you get lucky." She had found that the diagonal of a 3 by 4 rectangle was exactly 5 units, since the square built on this segment had an area of exactly 25 square units (by

[1] A "geoboard" is a square lattice of pegs upon which rubberbands can be placed to create polygons with lattice vertices. This simple tool is widely available in school classrooms. The most common size is a 25 peg geoboard arranged in a 5 by 5 square grid. This size is what the students here were given.

dissection). The most famous Pythagorean triple had been found and soon other multiples of it were discovered (like 6, 8, 10). Several students thought that this was a real freak of nature and assumed that (3,4,5) and its multiples were unusual loners. Unable to let this important historical moment pass by, I made the goal of the next class project the finding all such whole number diagonal lattice lengths under 100 (i.e. all Pythagorean triples or integers (a,b,c) such that $a^2 + b^2 = c^2$). I was well aware of the long cultural and historical role that this problem has played in mathematics for over four millennia. Many students thought this was a silly project since all they had to do was take integer multiples of (3,4,5) up to 100. The counter-example (5,12,13) soon emerged from early random searching.

In the next two sections of this paper I will describe how two students searched for Pythagorean triples. The two strategies employed were very different from each other and also quite different from anything that I have ever seen in my studies of the history of mathematics. These student investigations are individually fascinating from the standpoint of educational and cognitive research. They are also interesting in the way that each affected the direction of the other and of the rest of the class. Even more surprising is that both of them contain original mathematical ideas which seem to be absent from any existing mathematical literature. For this last reason I am violating the usual educational practice of using pseudonyms for subjects in educational research. The mathematical ideas are attributed to the students using their full actual names.

Both of these students entered the class with hostile and negative feelings about mathematics and both were initially glad that this was their last required course in mathematics before certifying as K-8 teachers. As we shall see, the nature of the class and their individual engagements brought about a variety of attitudinal and intellectual changes. Although the two detailed stories to be presented contain some unique mathematical ideas, the changes in attitude undergone by these students were not unique. A majority of students in the class underwent similar changes. Besides the written documentation of the projects, videotaped interviews were conducted with 15 different students. I shall return to these issues later.

Darron Saunders' Geometric Approach to Pythagorean Triples

In both high school and at the University Darron Saunders had struggled with mathematics and often his struggle had been very frustrating. Even when he did pass his classes he often came away feeling that he had learned nothing of any lasting value. Formal symbols manipulated in meaningless ways is how he summarized most of his secondary and University mathematics, and hence he avoided all but the minimum mathematics requirements. Saunders also has a long history of mild dyslexia which made all academic achievement more challenging for him. As a pre-service K-8 teacher he concentrated on special education because he has natural affinities for students with learning disabilities.

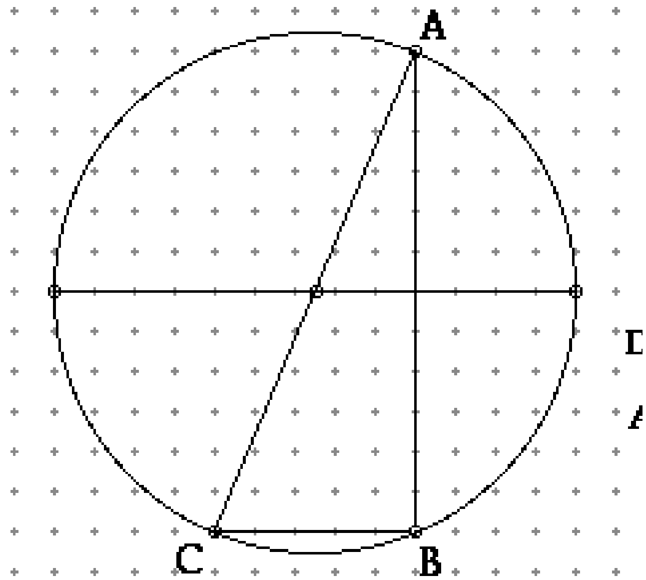
Like many dyslexics, Saunders has strong spatial and visualization abilities. These abilities are rarely rewarded in an algebra-dominated mathematics classroom. Saunders found outlets for his abilities mostly in the building trades and in sports. Projects that involved shapes, geometric ratios and scaling came easily to him and he particularly enjoyed working with a geoboard, graph paper, ruler, and compass.

When the search for Pythagorean triples emerged from the class's investigations of lattice geometry, Saunders was loathe to abandon physical geometry for number theory. While all other students in the class took up calculators, combined with tables and algebraic explorations, Saunders continued to search intensely for Pythagorean triples within the arena where they had originally emerged, lattice geometry. Saunders drew many pages of figures on graph paper using his ruler and compass. These figures were incomprehensible to other students in his group who ignored his investigation and went on with their numerical searches. I looked at more than 15 pages of Saunders' first drawings and I could not see where he was going. I gently remarked to him, "I think you're going to have to go numeric on this one." He ignored my remark and I did not pursue it remembering Maria Montessori's principle, "never interrupt a child during a period of absorption." Saunders is not a child but the principle seemed apt.

Eventually Saunders began giving new Pythagorean triples to students who were attempting to hunt them down with calculator searches. Saunders gave (18,15,17), and (20,21,29) to some students who thought that the all Pythagorean triples were integer multiples of either (3,4,5) or (5,12,13). Saunders' first written report contained many figures and few words of explanation and I could not at first understand what he was doing partly because as a historian of mathematics I was under the false impression that I knew all of the fundamentally different approaches to this problem. After a discussion with Saunders his method of generation became clear.

Using graph paper and a compass, he drew circles of integer diameter so that both ends of a diameter fall on lattice points in the same horizontal row. The center of the circle may or may not fall on a lattice point depending on whether the diameter is even of odd. He drew such a figure for each integer diameter separately. Next he examined each circle to see whether it hit any other lattice points. If it did, by symmetry, it hit four such lattice points, one in each quadrant. Connecting any three of these four symmetrical lattice points yielded a right triangle where all three sides have integer lengths. See Figures 1 and 2. This method depends on knowing that any right triangle inscribed in a circle will have a diameter as its hypotenuse. This geometrical concept is well known by carpenters who know how to use a metal square to draw a circle given the two endpoints of a diameter (i.e. pound in two nails at these endpoints and place the two legs of the square against the two nails, then slide the square with a pencil at the right angle).

Saunders' figures were an attempt to check for lattice points on each circle with an integer diameter under 100. He used different scales of graph paper and he began taping sheets of paper together to get larger sheets. He was well aware that care and accuracy were important here and his pencil-drawn figures were more clear and precise than the level of resolution of the computer-generated figures below.



Diameter (AC) = 13

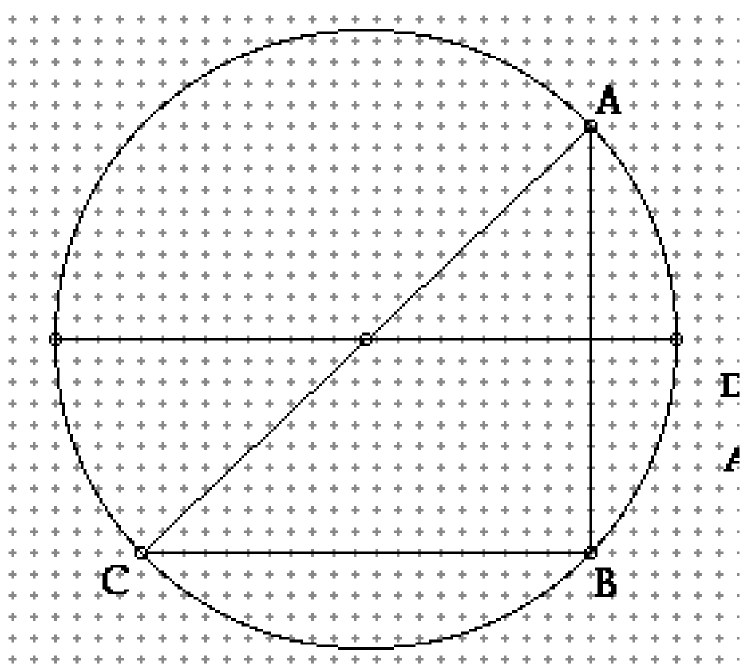
BC = 5

AB = 12

$AC^2 = 169$

$AB^2 + BC^2 = 169$

Figure 1- Triangle of Lattice Points on a Circle of Dimameter 13



Diameter (AC) = 29

BC = 21

AB = 20

$AC^2 = 841$

$AB^2 + BC^2 = 841$

Figure 2 - Triangle of Lattice Points on a Circle of Dimameter 29

Saunders was well aware that no matter how carefully he drew his figures some physical observational error might creep in especially as the diameters got larger. For this reason when he found a possible lattice point on one of the circles he would then check the resultant Pythagorean triple by calculation to make sure that the sum of the squares of the legs equaled the square on the hypotenuse. Occasionally he would have to discard a pseudo-Pythagorean triple where a circle seemed to hit a lattice point but the calculation showed that it was close but not quite there. See Figure 3 for an example of what seems to be another set of lattice points on the circle of diameter 29, but where calculation shows that it must be a near miss.

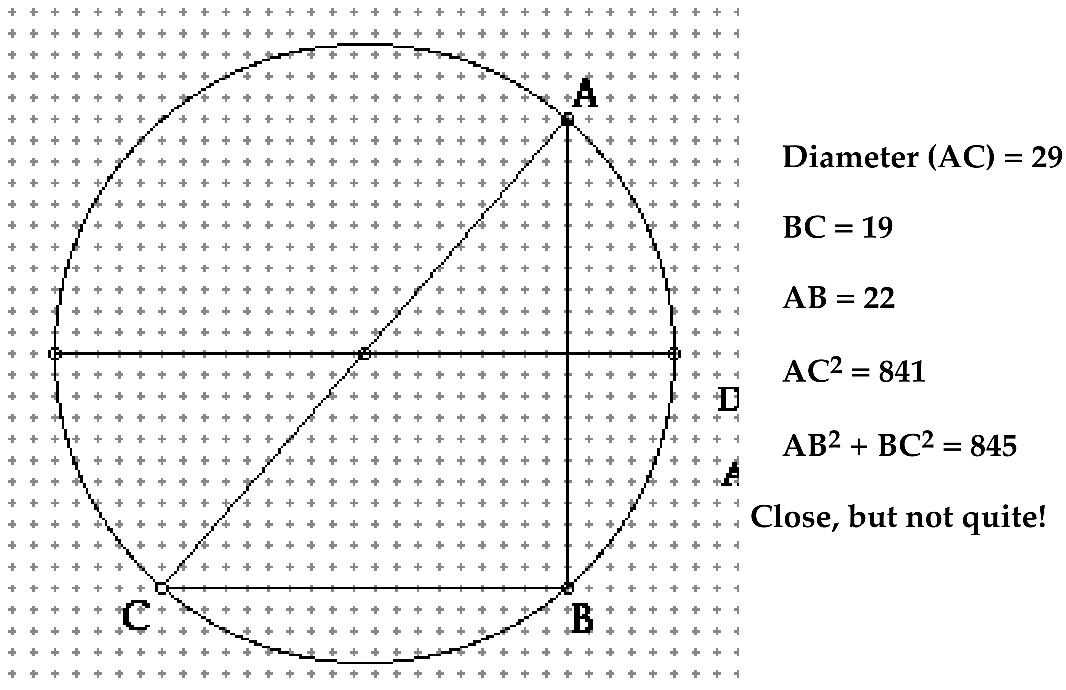


Figure 3 - Near Miss Lattice Points on a Circle of Diameter 29

The important feature of Saunders' method is that, although some pseudo-Pythagorean triples may turn up and have to be discarded, the method does not miss any true Pythagorean triples. It does systematically find all Pythagorean triples, but how can one be sure of this? This became the question that I posed for Saunders and other students who came to understand and appreciate his method. One thing that Saunders noticed is that all of the vertical legs (*AB*) of the triangles in his figures must have even integer lengths since they begin and end on lattice points and the horizontal diameter always bisects this leg at a lattice point. Saunders wondered about whether every Pythagorean triple must have at least one even leg. All of the examples found by the class always did. When I pressed him on this point he reasoned that even if a Pythagorean triple existed where both legs were odd, its double would have even integer legs and would eventually show up using his construction method. Therefore one must check each new Pythagorean triple for a possible common factor of two to make sure that the method generates all of them. Of course this never happened since it is true that all Pythagorean triples do have at least one even leg. Saunders was not interested in using his method to prove this number theoretic fact but only in arguing that his method is a fully general way to generate all Pythagorean triples.[2]

This again raises the issue of when a Pythagorean triple is primitive (i.e., the three numbers have no common factor). This was an important issue for those who were hunting for Pythagorean triples numerically. They knew that once they found a primitive one they could easily generate all of its multiples (similar figures to Saunders). As we shall see in the next section, this issue became crucial for another student who approached the problem through the use of table recursions.

Susanna Hernandez's Tabular Approach to Pythagorean Triples

Susanna Hernandez came to the class with many insecurities about her abilities to engage in mathematics. She said in an interview that she had often felt "dumb in math," although she did not generally consider herself a "dumb person" (see later section). Like most citizens of El Paso, Spanish is her first language but her English was generally better than many other students at UTEP. She plans to certify as a K-8 teacher specializing in bilingual education, and she hopes to teach at the second grade level. As this mathematics course developed she gradually became more and more enthusiastic. She found that the mathematics projects in the class were exciting to share with her extended Mexican-American family and this connection was influential on her view of the value of these projects. The study group in which she participated both in and out of class worked in Spanish and her increasingly influential position in this peer group was an important factor in her change in attitude about her own mathematical abilities. She said later that the Pythagorean triples project was the most important event in this evolution.

The first few projects in the class involved discrete counting projects where looking at patterns in the differences and ratios in consecutive terms in a sequence had proved a fruitful strategy for many students. These projects, however, had only involved making predictions in a single sequence of integers. Hernandez found these early projects much more to her liking than the latter more geometrical ones. She had developed a keen ability to find and understand recursive patterns, although her formal algebraic skills were poor. At the point in the class when the Pythagorean triples project emerged she gladly returned to a systematic hunt for integer patterns using recursive techniques, but this time she faced a triple sequence and had to devise a much more subtle recursive technique.

Several members of the class began this project by writing down a list of the first hundred perfect square numbers and then simply searching for any two that might add up to another member of the list. A number of Pythagorean triples were found this way, but Hernandez and her group soon realized that many of these were multiples of each other and that by finding a primitive triple they could then multiply it by integers and quickly generate its "family group." Thus for them the main problem became finding primitive Pythagorean triples.

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[2] I was originally convinced that Saunders' method did yield a proof of the fact that all Pythagorean triples have at least one even leg. Upon further reflection and discussion with other professional mathematicians, this is not all straightforward. The problem is that Saunders is always putting the centers of his circles on a lattice point or half-way between two points in the same row. One can not be sure, a priori, that in order to find all primitive Pythagorean triples as lattice points on the circle, one might not have to use the center of a square on the grid as a center for the circle.

The first three primitive Pythagorean triples that Hernandez's group found were (3,4,5), (5,12,13), and (7,24,25). Hernandez placed them in rows and noticed that the progression of the smallest numbers, 3, 5, 7, suggested an obvious arithmetic pattern. She also noticed that the largest numbers (the hypotenuses) were each one more than the corresponding largest legs. She next noticed that the largest legs increased by 12–4 = 8, and 24–12 = 12 = 8+4 , so she tried continue this pattern and generated Table 1, where the first column has a constant difference and the second column has a constant second difference and the third column is obtained by adding 1 to each entry in the second column. Continuing the pattern she generated triples and was pleased to find that they all satisfied the Pythagorean relation and that they were all primitive. Each new entry in the table could then be multiplied by any integer to generate a series of new families of Pythagorean triples.

Table 1 - Pythagorean Triples where c = b+1

a (Δ=2)	b (ΔΔ=4)	c (=b+1)	
3	4	5	**primitive**
5	12	13	**primitive**
7	24	25	**primitive**
9	40	41	**primitive**
11	60	61	**primitive**
13	84	85	**primitive**
15	112	113	**primitive**
17	144	145	**primitive**
19	180	181	**primitive**

Hernandez and her group thought that perhaps they might have found all of the Pythagorean triples since they now had an infinite recursive list of primitive triples. This hope was dashed by their classmates with the example of (8,15,17) which is primitive and does not appear in Table 1. With help of her fellow group member, Sonia Manzano, Hernandez went to work on another table. She thought that since her first table involved primitive Pythagorean triples using all of the odd numbers as legs (column a), perhaps the problem could be completed by generating a companion table where the first column consisted of the even numbers. She rearranged (3,4,5) into (4,3,5) and then doubled (3,4,5) to get (6,8,10). Putting these together with (8,15,17) she looked again for some pattern of recursive generation.

Hernandez found such a pattern as follows (See Table 2). She first noticed that the hypotenuses (c) were always 2 more than the largest legs (b). She then noticed that by adding $b+c$ in the first row it yielded the value of b in the second row (3+5 = 8). Although this does not work for moving from the second row to the third row it can be fixed up by subtracting the previous value of b, i.e. $b+c$–(previous b), or in Table 2, 8+10-3 =15, 15+17-8 = 24, 24+26-15 = 35, . . . Stated in the algebraic language of recursion she generated Table 2 by using the relations: $b_{n+1} = b_n+c_n-b_{n-1}$, $a_{n+1} = a_n + 2$, and $c_n = b_n+2$, although Hernandez herself never used such algebraic language. After checking that this method did indeed generate a table of Pythagorean triples, she later realized that she could also think of Table 2 in the same way that she had generated Table 1 where the first column has a constant first difference, and the second column has a constant second difference (i.e., the values of b go, up 5, up 7, up 9, etc.)

Hernandez noticed that only every other one of the rows in Table 2 is a primitive Pythagorean triple. This was not terribly surprising since she had used a double of (3,4,5) to get the recursion going in the first place. In fact the double of every row in Table 1 appears in Table 2 interspersed with an infinite list of new primitive Pythagorean triples. Hernandez now felt convinced that she could generate all Pythagorean triples by taking integer multiples of the primitives that occur in her two tables. After all she had now covered all possible cases of values of a, both odd and even, and no one in her group had any counter-examples to her theory. This was what I read in her first written report on this project.

Table 2 - Pythagorean Triples where c = b+2

a (Δ=2)	b (ΔΔ=2)	c (=b+2)	
4	3	5	**primitive (reversed)**
6	8	10	
8	15	17	**primitive**
10	24	26	
12	35	37	**primitive**
14	48	50	
16	63	65	**primitive**
18	80	82	
20	99	101	**primitive**

When I read Hernandez's report I was quite amazed in several ways. The Pythagorean triples in Table 1 are a special subfamily of all primitive Pythagorean triples that have a long history although I had never seen this tabular method used to generate them. This table made sense in many ways and helped me to rethink some issues in the history of ancient mathematics where all too often modern historians speculate on ancient methods using algebraic language that can obscure the original concepts. Table 2 was more surprising to me because I had never before seen any numerical or algebraic method that generated that particular list.

When this first round of reports was returned with comments, Hernandez presented her results and methods to the entire class. Most students were intrigued by her patterns of generation. Saunders and several others presented their work which included the example of (20, 21, 29) which is a primitive Pythagorean triple that does appear in either of Hernandez's two tables. Saunders was particularly impressed with her tables because of the ease with which with one can generate new triples and because they were so alien to his way of thinking. I added one comment of my own to the discussion which was that Table 1 could have started with the flat, degenerate "triangle" (1,0,1) and likewise Table 2 could have started with (2,0,2). I pointed out that these flat triangles fit perfectly well into the recursion patterns that Hernandez was using. Saunders pointed out that even if we considered these flat triples as degenerate triangles they still did not fit with the original search for integer diagonals on the geoboard, since these did not involve any real diagonals. It was only during this class discussion that I came to fully understand Saunders' method of constructing triples.

While waiting for the next class meeting I could not resist doing my own round of intensive research since I had just been shown by my students two methods for generating Pythagorean triples which I had never seen before. A search through standard works in number theory and history of mathematics revealed no mention of either Saunders' or Hernandez's methods. Oddly enough the only connection that made any real sense was between Hernandez's tables and the geometric construction of Pythagorean triples given by Euclid in Book 10, Lemma 1 before Proposition 29. Euclid's construction can be viewed as using the parameter of a constant difference between one leg and the hypotenuse.

When the class met again we continued our full class discussion and began constructing Table 3. Observing that (20,21,29) had a difference of 8 between the large leg and the hypotenuse, we looked for others with that difference. Following Hernandez's method we found others by either multiplying previous triples or by rearranging them. (5,12,13) could be written as (12,5,13) and any triple in Table 2 could be multiplied by 4. This was enough to get us started on a pattern which led to Table 3.

Table 3 - Pythagorean Triples where $c = b+8$

a (Δ=4)	b (ΔΔ=2)	c (=b+8)	
8	0	8	
12	5	13	primitive (reversed)
16	12	20	
20	21	29	primitive
24	32	40	
28	45	53	primitive
32	60	68	
36	77	85	primitive
40	96	104	
44	117	125	primitive
48	140	148	

Saunders' continuing ruler and compass search had led him to find (33,56,65). This led the class to construct Table 4 where the hypotenuses are all 9 more than the second legs. Again the pattern was started by multiplying rows in Table 1 by 9 and by rearranging other primitive triples like (8,15,17) and (20,21,29) into (15,8,17) and (21,20,29). In this table we were all surprised to find that instead of finding an alternating pattern of primitives and multiples we found that there were two primitive triples between multiples from Table 1. After the first two reversed primitive rows, all of the others were new primitive triples.

Table 4 - Pythagorean Triples where c = b+9

a (Δ=6)	b (ΔΔ=4)	c (=b+9)	
9	0	9	
15	8	17	**primitive (reversed)**
21	20	29	**primitive (reversed)**
27	36	45	
33	56	65	**primitive**
39	80	89	**primitive**
45	108	117	
51	140	149	**primitive**
57	176	185	**primitive**
63	216	225	
69	260	269	**primitive**
75	308	317	**primitive**

The whole class was able to follow most of the details of this investigation up to this point. This became evident in their rewritten reports where recursive tabular organization became the most popular explanation. An important issue that was made clear by Hernandez was that only three rows in a table are needed to get the recursion going in the second column, but that it was important to check that it was indeed generating true Pythagorean triples. The initial three rows had to have a certain structure in order for this to work out (i.e., an arithmetic sequence in the first column). After extrapolating the assumed constant second differences in the second and third columns the method could be checked for valid generation of triples.

Many were particularly interested because I made it clear that these ideas were entirely new to me and that they had come from students who had started the class with no prior love of mathematics or special expertise. There were several students in the class who were pursuing K-8 certification with a specialization in mathematics. Generally these students had much better algebraic skills and often attempted to solve all problems algebraically. At this point none of them had achieved a complete algebraic solution, although several were able to do so later with minimal hints. The usual algebraic solution in the form ($2uv$, $u^2–v^2$, u^2+v^2) was described by several of these students as awkward and inefficient to actually use since it frequently gave multiples and conditions on u and v which guaranteed primitive triples were difficult for them to find and explain.

I next provided a brief historical presentation where I showed the class an ancient Babylonian tablet (Plimpton 322) with a list of Pythagorean triples that included some very large primitive ones (Katz, 1993). This interested many students who enjoyed seeing the base 60 system. I also presented Euclid's construction and discussed its relations to Hernandez's tables. This had far less impact on the class because those who most enjoyed Hernandez's method usually wanted to avoid geometry as much as possible. Euclid's method is far less direct than Saunders' and even Saunders' geometrical method was not popular with the class.

The class as a whole did not make any more tables, but for those who wanted to use this method to finish the original task of finding all Pythagorean triples under 100, two more tables were needed. They can be generated in the same manner by first looking at possible reversed triples in order to see where the next new primitives will occur and then putting in multiples of entries from previous tables. Reversing (7,24,25) from Table 1 shows that the next place to look is when $c = b+18$. Half of the rows in Table 5 come from doubling the rows in Table 4. In Table 5, however, the other half of the rows are not all primitive. Every third entry is 9 times one of the rows in Table 2. Thus the pattern of primitives here is a bit more complicated since this is the first time that the constant difference between c and b has two distinct prime factors.

Table 5 - Pythagorean Triples where c = b+18

a (Δ=6)	b (ΔΔ=2)	c (=b+18)	
18	0	18	
24	7	25	**primitive (reversed)**
30	16	34	
36	27	45	**9 x (4,3,5)**
42	40	58	
48	55	73	**primitive**
54	72	90	
60	91	109	**primitive**
66	112	130	
72	135	153	**9 x (8,15,17)**
78	160	178	
84	187	205	**primitive**
90	216	234	
96	247	265	**primitive**

The last table needed to complete the project using Hernandez's method is Table 6 where $c = b+25$. Every fifth entry in this table is 25 times one of the rows in Table 1, while all of the others are primitive. The first three are reversed rows from Tables 2, 3, and 5 respectively. This last table adds only one last triple to the complete list of those under 100. This triple, (65,72,97), was the one most often missing from student reports on this project.

Table 6 - Pythagorean Triples where c = b+25

a (Δ=10)	b (ΔΔ=4)	c (=b+25)	
25	0	25	
35	12	37	**primitive (reversed)**
45	28	53	**primitive (reversed)**
55	48	73	**primitive (reversed)**
65	72	97	**primitive**
75	100	125	
85	132	157	**primitive**
95	168	193	**primitive**
105	208	233	**primitive**
115	252	277	**primitive**
125	300	325	
135	352	377	**primitive**

It turns out that in one sense Hernandez was correct when she said that all of the primitive triples can be found from the first two tables. Although primitive triples exist that are not in Tables 1 or 2, by reversing primitive triples from Tables 1 and 2 one finds all possible new differences between b and c that can occur in any primitive triple. Thus by looking at the first two tables one knows in advance where all the other table recursions occur which produce new primitive triples. This will be proved in the next section.

Mathematical Implications of the Student Investigations

This section will discuss a variety of mathematical ideas and concepts concerning Pythagorean triples that arose from my interactions with the students in this pilot class. I will give some modern proofs of a variety of results that suggested themselves either directly or indirectly from the investigations of Darron Saunders and Susanna Hernandez. These ideas came about from my own meditations on the relations between the history of mathematics and the activities of the students in this investigation. Most of these thoughts did not become clear in my mind until several months after the class ended. By that time I had largely lost contact with these former students whose lives became involved with the public schools. Although these ideas were clarified out of my own thoughts as a mathematician and educator and through discussions with other professional mathematicians and educators, it must always be remembered that none of this would have been possible without the original creative insights of the students in the project, particularly the work of Saunders and Hernandez.

Let us first consider geometrical approaches to the construction of Pythagorean triples. The method used by Saunders is simple, direct and depends only on the fact that right triangles can be inscribed in a circle if and only if the hypotenuse is a diameter. This is one of the oldest and most fundamental concepts in geometry and using this principle along with a grid to search for Pythagorean triples seems so basic that I was surprised that this idea is not mentioned in any mathematical sources that I know of. It seems that this idea must surely have been explored in ancient times, possibly in several independent instances (Gnaedinger, 1996). Perhaps it does not exist in mathematical literature because it contains elements of physical empiricism and lacks the usual abstract character of a "proof." Direct scientific investigations with physical tools are the most fruitful sources of mathematical concepts (von Neumann, 1984), but Platonic trends in the history of mathematics have tended to erase them. The elimination of such tool-based investigations from mathematics classrooms serves to alienate many students from further pursuits and even those who do continue tend to have constricted abilities to apply what they learn (Arnol'd, 1990; Dennis, 1995; Dennis & Confrey, in press).

There are other geometrical constructions of Pythagorean triples that are less empirical, but they depend on much more complicated geometrical principles, e.g. the construction of Richard Vogeler (Conway and Guy, 1996, p.172). Let us turn next to Euclid's method for generating Pythagorean triples, given in the Elements, Book X, Lemma 1 before Prop. 29 (Heath, 1956). Euclid considers the problem of when the difference of two integer squares is also an integer square. As usual, he considers a gnomon figure (Fig. 4) and looks at the difference in area between *DBFE* and *HIGE*. Rearranging this strip of area into a rectangle *ABLK* (i.e. *ADHK* = *LFGI*), he seeks conditions that will insure that this integer rectangle will also be an integer square. This will happen if and only if the integers BL and KL are similar plane numbers. By this he means that BL and KL are numbers that represent the areas of similar rectangles. Thus these two lengths can be factored as $BL = ab$, and $KL = cd$, where $a/b = c/d$, which implies that $ad = bc$, and hence abcd is a perfect square. There is one additional condition that is required for this construction to work and that is that $KH = HI$ must be an integer which means that difference $KL - BL$ must be even. Euclid insures this by stating that the two similar plane numbers must be either both even or both odd.

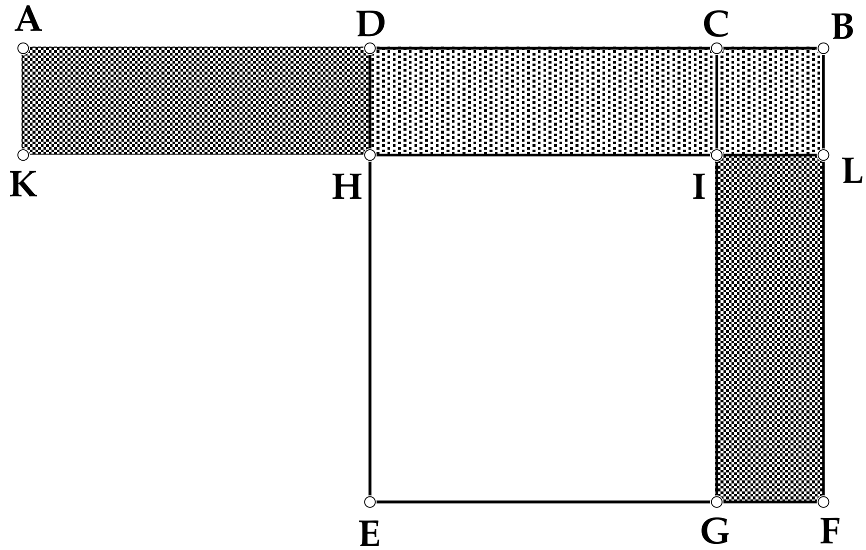

Figure 4 - Euclid's Construction of Pythagorean Triples

The static and passive style of Euclid's exposition can obscure the way in which his construction actually generates Pythagorean triples. In practice it does yield an efficient way to generate triples which is closely connected to the method of Susanna Hernandez. One can start from any pair of similar plane numbers, s and t (both odd, or both even), where $BL = t$, $KL = s$, hence $HI = (s-t)/2$, and $HL = (s-t)/2 + t = (s+t)/2$ thereby generating the triple ($\sqrt{st}$, $(s-t)/2$, $(s+t)/2$). How might this actually work in practice? The simplest way seems to be to choose the smaller of the similar plane numbers first. For example let $t = 1$. Since t is a square with only the trivial factorization, the only similar plane numbers are other squares and they must be odd. Letting s range over all odd squares yields Hernandez's Table 1. Next let $t = 2$. Now the only similar plane numbers are $s = (2x)x$, for any integer x, so s can range over the doubles of all perfect squares. This yields Hernandez's Table 2. Since t is the difference between the hypotenuse and one of the legs, fixing different values of t will continue to generate these tables. To generate Table 3, let $t = 8$. This time there are two possible factorizations, $t = 1{\cdot}8$, and $t = 2{\cdot}4$. Letting s be a similar plane number in the first sense means $s = (8x)x$, and these values of s yield the non-primitive entries in Table 3, while letting s be similar in the second sense means that $s = (2x)x$, and this yields the primitive entries in Table 3.

To what extent is Euclid's method truly geometrical? Saunders' method is much more directly related to the finding of Pythagorean triples as right triangles in the geometrical sense. It could be argued that Saunders' method is exactly the kind of practical, physical, tool-based approach that Euclid was trying so hard to avoid, since the his Platonic tradition strictly separated mathematics from practical science. Euclid's method however runs into difficulties within his own epistemological structure. BL and

KL are line segments or lengths, hence one dimensional, but it is essential in the construction that they be "similar plane numbers" which means that one must consider them as the areas of similar rectangles in order to find appropriate values which will make the construction work. On the surface this violates the dimensionality of the construction; something that elsewhere Euclid is loathe to do. There is no easy way to preserve dimensional integrity here although there are clever and tedious ways to linguistically sidestep this dimensional difficulty. This issue is discussed again and again in the history of mathematics by Descartes, Dedkind and others (Fowler, 1992). However, the only way to efficiently make use of this construction as it stands is to find the similar plane numbers by thinking of them first as areas of similar rectangles, and then to use these areas (given as products) as lengths in order to construct a triple as in Figure 4.

Historians have intensely debated the extent to which Euclid's Elements are truly geometrical (van der Waerden, 1976; Unguru, 1976). Some, like van der Waerden, have argued that much of Euclid is "geometrical algebra" that has strong roots in the arithmetical traditions of Babylon. After reflecting on the work of my students and possible relations to history, I am inclined to believe that in the case of Pythagorean triples, Euclid's method has strong ties to recursive tables and less connection to geometrical action. This seems to be an example of van der Waerden's claim that the Greeks gave Babylonian mathematics geometrical attire. I do not claim, however, that this implies an entirely algebraic view. Recursive techniques for generating tables are not at all equivalent to algebraic methods in the strictest modern sense. The tables generated by Susanna Hernandez were created numerically using first and second constant differences vertically in each column. This pattern could be written down in algebraic language, but such language would obscure the original conception and would not make the generation of the tables any easier. Quite the contrary, working either by hand or with a computer, the tables are most easily generated by using the original recursive techniques.

The tables of Pythagorean triples generated by Hernandez lead me to speculate that, for several reasons, her method is a more plausible explanation of how Babylonian tables like Plimpton 322 might have been generated. First, this method stays within a tabular representation and is quite easy to generate using only simple arithmetic. Second, each of the tables is ordered with respect to the angles in the triangles, that is to say the angle subtended by the first leg decreases while the angle subtended by the other leg increases. This is easily seen since the first leg is growing arithmetically while the second leg is growing quadratically. The famous tablet, Plimpton 322 is arranged by angle and thought to have served a trigonometric function. To create such a list one would have to take entries from different recursively generated tables and merge and sort them by angle (or ratio of sides). Having each separate table already ordered by angle would certainly be a great aid in such a task. Historians have offered possible algebraic explanations but these can be difficult to see in the original system of mathematical representation, i.e. tables. It seems more plausible to make such speculations by working strictly within the original representational form.

Another mathematical issue that arose in this project, was the trigonometric use of Pythagorean triples. Using the circle and lattice construction, Saunders became aware that he could find Pythagorean triangles almost anywhere on the circle. Using only the first few recursive tables, Hernandez saw that the Pythagorean triples bunch up right away at small angles and at angles near 90°, but more tables are needed in order to find more angles around 45°. It seems certain that ancient mathematicians in several cultures made use of Pythagorean triples to study the proportions in right triangles and their relations to angles (Katz, 1993; Gnaedinger, 1996). The concept of angle itself may have been shaped from such investigations. Babylonian tablets like Plimpton 322 empirically demonstrate that by using large Pythagorean triples one can find integer right triangles that are arbitrarily close to any desired angle.

In modern mathematical language one would say that Pythagorean triples are dense with respect to angle. When I searched modern mathematical literature for this important theorem I was surprised that I was unable to find it. It is indirectly implied by some advanced theory, but never directly stated. This led me to seek an elementary demonstration of this important historical theorem. An idea for a simple proof came from my colleague James Nymann, who has discussed Pythagorean triples in his number theory courses for many years but had never heard of this theorem.

Consider the usual algebraic equations for generating Pythagorean triples ($2uv$, u^2-v^2, u^2+v^2) which is primitive if u and v are relatively prime and not both odd. This formulation occurs indirectly in Diophantus and in modern algebraic language in every book on number theory since Fermat (for a derivation see, for example, Conway & Guy, 1996). Consider the two similar triangles in Figure 5. By similarity:

$$\frac{\sec(\alpha)+\tan(\alpha)}{1} = \frac{(u^2+v^2)+(u^2-v^2)}{2uv} = \frac{u}{v}\ .$$

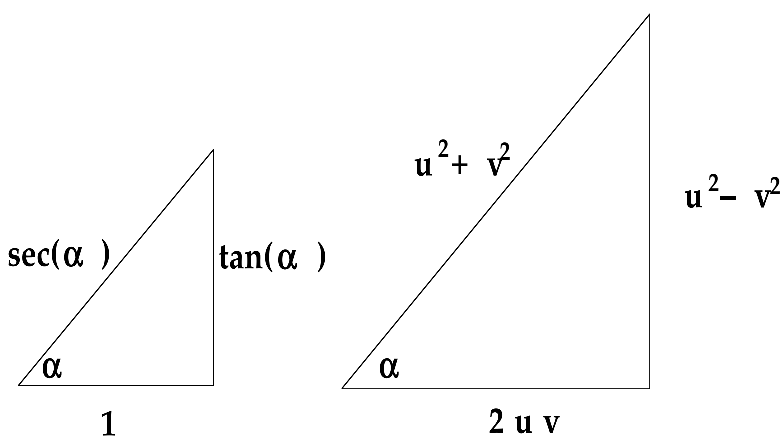

Figure 5 - Proof of the Density of Pythagorean Triples with Respect to Angle

Given any angle, α, any sequence of rational numbers converging to $\sec(\alpha)+\tan(\alpha)$ will produce a sequences of pairs (u,v) that generate a sequence of Pythagorean triples with angles that converge to α.

At the end of the last section it was pointed out that by reversing the entries in only Tables 1 and 2, one knows in advance exactly where to find all new tables that yield new primitive triples. This can be seen by looking at the two possible differences between the hypotenuse and a leg:

$$(u^2+v^2)-(u^2-v^2) = 2v^2, \text{ and } (u^2+v^2)-2uv = (u-v)^2\ .$$

Here one sees that these differences must either be twice a square or an odd square (since u and v are not both odd, $u-v$ is odd). Looking at $c-b$ in Table 1 gives all the doubles of squares, and looking at $c-b$ for the primitive entries in Table 2 gives all odd squares. Once one primitive Pythagorean triple with a certain difference exists then the recursion will produce an infinite list (left to the reader). In this sense Hernandez's instincts were correct that Tables 1 and 2 are the key to the entire problem.

This algebraic characterization of Pythagorean triples lends itself nicely to giving simple proofs, but its separation from other direct generating activities leads one away from concepts like geometric density. The closed-form algebraic solution to the problem, which has taken precedence over all other approaches in modern books, has constricted the mathematical breadth of activities that are discussed. Even though Pythagorean triples are no longer used for trigonometric purposes, every mathematician to whom I have mentioned these recursion and density issues was fascinated and surprised that they are not mentioned in modern books. They provide the kind of cross connections that most teachers strive to bring into the classroom.

Implications for Educational Theory

In recent years researchers in mathematics education have been striving to form a more viable theory for how students develop advanced mathematical thinking. Focus on the function concept provided some general agreement in the last decade, but a more detailed theoretical discussion is now taking place. One increasingly popular theory being proposed is the idea of reification (Sfard, 1992; 1994). Briefly put, reification theory proposes that in order to attain advanced mathematical thinking students must transform activities and processes into objects. For example, a function may initially be seen as arising from some action or process but students must come to see it as a mathematical objects which can be structurally manipulated in a higher abstract situation. Failure to reify, that is to make processes into objects, is seen by Sfard as an epistemological obstacle which prevents students from reaching higher levels of mathematical abstraction.

Certain aspects of the history of mathematics provide important examples of how reification has led to important advances in mathematics, however Sfard has proposed reification as a central metaphor for identifying obstacles which impede the mathematical thinking of students. As an educational theory reification tends to reinforce a hierarchical view of mathematics with the highest value being placed on symbolic algebraic representation and the structures that result from such formalisms. In this sense reification theory has come under attack by Confrey and Costa (in press). Their critique emphasizes the view that a robust mathematics must provide a wide variety of tools for the modeling of diverse situations and that reification does not encompass this wider mathematical activity. Hence reification as a central metaphor is an inadequate educational theory which may in many cases may even curtail the development of student thinking by limiting the scope of what is deemed appropriate mathematical activity.

In one sense the original concerns of these theorists have been different. Sfard and several of her supporters (e.g. Dubinsky) have studied the successes and failures of students in advanced mathematics courses designed primarily for university students majoring in mathematics. Confrey's research examined students at many levels who use mathematics in wide variety of scientific and engineering settings. For this much larger population of students a central metaphor of modeling and tools is far more appropriate. A simple separation of educational theory in mathematics into "pure" and "applied" would be a grave mistake. The connections between tables, geometry, and algebraic structure remains of fundamental concern throughout all of mathematics and such connections can have the flavor of modeling at even the most advanced levels. An even more disturbing trend is to see reification theory applied in educational research at an elementary level where linear and quadratic functions are first introduced. At the recent 1997 meetings of the American Educational Research Association (AERA) several presentations discussed how to quickly wean elementary algebra students away from tables and diagrams in favor of symbolic algebra. These researchers used reification theory as a kind of stage theory which viewed tables, recursion and geometric constructions as epistemological obstacles.

The student investigations of Saunders and Hernandez present difficulties for reification theory. The excitement, originality and profound intellectual content of their work fundamentally depended on their commitments to forms of mathematical representation that reification theory would classify as "obstacles." If Pythagorean triples must be generated with a symbolic function like, $F(u,v) = (2uv, u^2{-}v^2, u^2{+}v^2)$, and this function must be seen as a mathematical object in order to gain advanced mathematical understanding, then all of the ideas in previous sections are lost because they depend upon close attention to process and action. Indeed, these ideas have probably disappeared from mathematical literature because reification has been acting for many years as a force that transforms the historical record of mathematics (Arnol'd, 1990). Despite Sfard's claims to the contrary, Confrey and Costa point out that reification theory has progressive absolutist underpinnings in that it assumes that since objectification of algebraic entities has led to progress in the past, this must always be the best way to go at any level. In contrast, the tool-modeling metaphor encourages a broader range of forms and allows students a freer range of possible expression.

Piaget's theory of genetic epistemology suggests that we can gain profound insights into human cognition by studying the historical genesis of ideas (Piaget & Garcia, 1989). In the case of Saunders and Hernandez one can see both genetic epistemology and its reversal at work. The problems, projects and classroom environment were designed with the help of research in the history of mathematics, but the work of these two students also led to a more profound understanding of mathematical history. Genetic epistemology can be a two way street when one listens closely to students. This new depth of historical understanding suggests ways to further develop student projects and a classroom environment which will make creative and diverse mathematical expression by students even more likely. Everything old is new again, and again, and again.

Impact on Mathematics Classroom Environment

This paper has focused on two students and one project in order to convey in depth the kind of mathematical experience that can be had by K-8 preservice teachers given appropriate curriculum, assessment and classroom environment. Although the insights of Saunders and Hernandez in this project were particularly original they were not atypical students. Many students entered the class with better preparation and better attitudes about mathematics. By the end, nearly every student in the class found several different projects which inspired them to go beyond the required level of the class. This was evident in their project reports, their final essays, class evaluations, and in videotaped interviews. In the past traditional versions of this class often have had 50% failure rates and been pervaded by a cynical attitude of "just tell me what I have to do to get out of here."

The next phase of the reform faced the difficult issue of how to bring this kind of experience routinely to all preservice K-8 teachers. The effort that I put into this initial pilot class went way beyond what a professor normally devotes to a three credit, one semester class. At a university with a teaching load of three courses per semester, other members of the mathematics department were justifiably skeptical as to whether this kind of teaching could become the norm even for this one class. On the other hand several of my colleagues saw both the possibility and importance of insuring that every K-8 teacher that we graduate leaves the university with at least one semester's engagement with creative mathematics. With the generous support of the National Science Foundation, we set out to achieve this goal.[3]

During the initial pilot all of the models, manipulatives and materials (e.g. geoboards) used in the class had to carried back and forth on a daily basis. Some distant computer labs were available to the students for word processing but no innovative computer software was available. These problems were solved by the creation of a dedicated classroom fully equipped with a wide variety of models and materials and with 30 new PowerMac computers with direct internet access. The materials are permanently stored in the classroom so that they are constantly available for spontaneous use. The computers are all set up along two outside walls in the room with the center of the room filled with tables and chairs to foster group discussions. One end of the classroom is equipped with both a chalkboard and a high quality computer projection system. The room does not look or feel like a "computer lab" but is instead a discussion-oriented room where computers are available.

The question then arose as to what computer software would best facilitate student mathematical investigations. The work of Saunders and Hernandez illustrate how differently students can approach the same problem and we wanted computer software that would expand rather than restrict the range of possible expression. *Geometer's Sketchpad* was installed in order to allow students to quickly and directly extend and record explorations that involve actions with a geoboard, ruler, compass or graph paper . On the other hand, we installed *Function Probe* which allows students to quickly and efficiently explore difference patterns in multiple sequences in a tabular setting. *Function Probe* also has all of the features of the best graphing calculators available in a format that is easy for students to access. No one software program would allow for this flexibility and so these two softwares are now the main exploratory tools that are available to all students in the class. Thus far they have provided enough flexibility to aid most students while not being so complex that their introduction becomes a huge part of the course. More advanced softwares such as *Mathematica* or *Maple* seemed entirely inappropriate in this setting. Some professors have even questioned the use of *Geometer's Sketchpad* and *Function Probe* as being perhaps too complex. This is a topic for further study as our reform proceeds, but in preliminary studies students have found these two softwares to be useful and effective when combined with up-to- date word processing, all on the same machine.

In order to implement the new curriculum and fully utilize the new classroom and equipment we have hired undergraduate students to work as interns. These interns are students who are enrolled in teacher education programs either as mathematics majors or as K-8 teachers with a mathematics specialization. These interns keep the new room open outside of class time and help the students to prepare their reports by listening to their thoughts and asking questions which will lead to further articulation. Experienced interns along with graduate students in the M.A.T. program now help to read and comment on initial student reports. All rewritten student reports

[3] Our support comes through a preservice teacher collaborative known in El Paso as the Partnership for Excellence in Teacher Education (PETE).

are read by the professors. Training of the student interns involves familiarizing them with new assessment concepts and with the use of models and software. In this way the professor's work load in the course is brought within reasonable bounds while still involving all students in an exciting mathematical experience. Our goal is to have a core of mathematics professors and a steady stream of experienced student interns who will sustain a permanent course reform by the end of our five year NSF grant (we are entering our third year).

Another wider aspect of our reform is our intention to link this course with field work in the schools. This mathematics course is now a required part of a three semester block program in the College of Education. Students will all be taking this course simultaneously with their initial field work in the schools. One professor of education already requires the adoption and implementation of at least one of these mathematics projects in a public school setting as part of her curriculum course. Many students in this initial group have entered public school classrooms with a greatly expanded view of the mathematical abilities of children and a consequent desire to expand their own mathematical experience (reported from fieldwork in the College of Education). Further studies are underway of the benefits and possibilities of such an integrated program. Such studies involve testing, surveys, videotaped interviews, and classroom videotapes. Ultimately we would like to know what impact the university can have on the classroom environment of school children. Will children and teachers be given the tools and opportunities for expanded mathematical expression? Will future teachers listen more intelligently to the mathematical expressions of children?

Impact on Student Attitudes Towards Mathematics

Preliminary evidence indicates that the reform of just this one mathematics class is having a significant impact on the attitudes of many students in the K-8 teacher education program. Further analysis will be forthcoming from surveys, interviews, and testing, but even all of these in-house studies do not determine what matters most; can a different university mathematical experience change the way future teachers will behave once they are in their own classrooms? Difficult and expensive longitudinal studies are needed for this and solid results can be a long time in coming. In the meantime the mathematics department must make many immediate decisions based on intelligent use of cognitive theory and pilot experiences. Since this paper is a pilot report, in closing, let me return to the two students on whom I have focused. Unique as their work on the triples project was, their attitudes upon entering the class were very typical. They both viewed mathematics with fear and loathing. They were both glad that this course was their last mathematics requirement, and they both had the initial attitude of "let's get it over with." Information was collected from the student portfolios which included initial and final essays on the question "What is mathematics?" and from videotaped interviews conducted midway through the class with about half the class (including Saunders and Hernandez).

Darron Saunders was an early convert to the philosophy of the class. By the third week he was an enthusiastic participant and a de facto group leader in the investigations. When the investigations turned towards geometry in the fifth week his work became obsessive (his own description). When he was interviewed during the eighth week, he described his high school and university experiences as being painful and lacking in any experiences that might have sparked his interest or provided any depth of understanding. He passed his courses but he said, "I was completely ready to give up math forever . . . I wasn't getting anything out of it . . . it was kind of like learning the grammar of another language that made no sense. . . I've already learned more here than in all of my other (university math) classes."

Saunders then told of his surprise and elation at being sought after outside of class by other students who wanted to hear his mathematical ideas. Being a relaxed and sociable person he listened well communicated easily but he was shocked at being asked to give precise opinions concerning mathematical concepts. He was most surprised at seeing many different approaches that could all yield important ideas in a mathematical investigation. It became an intense form of social interaction which captivated him. Although it seems silly to him now, he described his former belief that there exists only one correct approach to any mathematical problem. Saunders then described his change of attitude.

> I see mathematics now as logic and reasoning. . . as a way of thinking and analyzing. . . I see numbers now as a tool, like a compass. . . they're not something I want to shy away from anymore. . . I want to go on with this, who knows how far? . . . I feel somewhat cheated. Why wasn't I taught mathematics like this much earlier? Why was this the last required mathematics class instead of the first? If I had started like this who knows where I would be now with mathematics! . . . If there were more mathematics classes taught in this way I would gladly take more than the minimum requirement.

Saunders, like many teacher candidates, had postponed his mathematics courses until his senior year. He graduated with a K-8 teaching certificate in special education, but when he found his first job in the Fall of 1996 it was not in his specialty. He specifically asked for and got a job that focused on teaching sixth grade mathematics and science in a middle school in a very low income area. He knows that his background is deficient to achieve all that he wants, but he is confident that he can now see a way to continue to develop his skills and knowledge. This new found confidence he attributes to his engagement and achievements with the mathematics projects in the pilot course.

Susanna Hernandez's early experiences in the class were cautious and skeptical. Her specialty is in bilingual education and Spanish is her first language. Being able to work in a group with other Spanish speaking women was an advantage for her. The rewrite policy was also important for her and she developed increasingly creative and articulate forms of expression in her reports. She did not begin speaking out strongly in class until around the eighth week of the class. This was when the Pythagorean triples project began and her engagement led to a new level of confidence and achievement. She was interviewed on videotape just as that project was going on in class. Here are some verbal comments that she made at that time.

> I'm not big on math . . . but . . . in this class I've seen a kind of never-ending learning that I never knew existed.

> It can be very time consuming. . . I get into it so much that it can go on and on. . . but these problems are interesting. . . so interesting to me that I show them to my family, and to my nephews and they say ohhhh that's cool but did you think about this. . . they find it interesting and I think it's fun to share what I'm learning with my family.

> I'm not a real creative type of a person but group work helps me to see the big picture when I have ideas from others.
>
> I'm learning about myself in this class. I'm learning that I'm not as dumb in math as I thought I was. I used to think Oiii . . . math no. . . I hate it. . . but now I think anything's possible. I can do it.
>
> You don't just turn it in once and then get back right or wrong. You get it back with comments and you keep thinking about. . . oh I guess I could have done this or this and then we add stuff on and get a little deeper into it. Anything might become useful in the next problem. You're not just submitting something and forgetting about it.
>
> This class has allowed me to relax with math and to try this and that. . . and. . . NO LIMIT!

In her final written essay at the end of the class, Susanna sums up her experiences as follows.

> My attitude towards math was changed after working on the Pythagorean triples project. I didn't see myself as much of a math person. After finding my triple results it changed my self esteem as far as math goes. When I was working on the problem and started finding patterns for the triples I felt some kind of high or like my adrenaline was really pumping every time I would find a different one. I did learn a lot about math in this class. . . and about things that I never imagined as being mathematical, but what I learned most was about myself. I found myself looking forward to getting my graded projects back and feeling really good about sitting in class and absorbing all kinds of ideas from all kinds of people. . . . I would really recommend more classes like this course.

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