ARTICLES

René Descartes' Curve-Drawing Devices: Experiments in the Relations Between Mechanical Motion and Symbolic Language

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Introduction

By the beginning of the seventeenth century it had become possible to represent a wide variety of arithmetic concepts and relationships in the newly evolved language of symbolic algebra **[19]**. Geometry, however, held a preeminent position as an older and far more trusted form of mathematics. Throughout the scientific revolution geometry continued to be thought of as the primary and most reliable form of mathematics, but a continuing series of investigations took place that examined the extent to which algebra and geometry might be compatible. These experiments in compatibility were quite opposite from most of the ancient classics. Euclid, for example, describes in Book 8–10 of the *Elements* a number of important theorems of number theory cloaked awkwardly in a geometrical representation **[16]**.[1] The experiments of the seventeenth century, conversely, probed the possibilities of representing geometrical concepts and constructions in the language of symbolic algebra. To what extent could it be done? Would contradictions emerge if one moved freely back and forth between geometric and algebraic representations?

Questions of appropriate forms of representation dominated the intellectual activities of seventeenth century Europe, not just in science and mathematics but perhaps even more pervasively in religious, political, legal, and philosophical discussions **[13, 24, 25]**. Seen in the context of this social history it is not surprising that mathematicians like René Descartes and G. W. von Leibniz would have seen their new symbolic mathematical representations in the context of their extensive philosophical works. Descartes' *Geometry* **[11]** was originally published as an appendix to his larger philosophical work, the *Discourse on Method.* Conversely, political thinkers like Thomas Hobbes commented extensively on the latest developments in physics and mathematics **[25, 4]**. Questions of the appropriate forms of scientific symbolism and discourse were seen as closely connected to questions about the construction of the new apparatuses of the modern state. This is particularly evident, for example, in the work of the physicist Robert Boyle **[25]**.

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[1] See, for example, Book 10, Lemma 1 before Prop. 29, where Euclid generates all Pythagorean triples geometrically even though he violates the dimensional integrity of his argument. Areas, in the form of "similar plane numbers," are multiplied by areas to yield areas. There seems to be no way to reconcile dimension and still obtain the result.

This paper will investigate in detail two of the curve-drawing constructions from the *Geometry* of Descartes in such a way as to highlight the issue of the coordination of multiple representations (see, e.g., **[6]**). The profound impact of Descartes' mathematics was rooted in the bold and fluid ways in which he shifted between geometrical and algebraic forms of representation, demonstrating the compatibility of these seemingly separate forms of expression. Descartes is touted to students today as the originator of analytic geometry, but nowhere in the *Geometry* did he ever graph an equation. Curves were constructed from geometrical actions, many of which were pictured as mechanical apparatuses. After curves had been drawn Descartes introduced coordinates and then analyzed the curve-drawing actions in order to arrive at an equation that represented the curve. Equations did not create curves; curves gave rise to equations.[2] Descartes used equations to create a taxonomy of curves **[20]**.

It can be difficult for a person well schooled in modern mathematics to enter into and appreciate the philosophical and linguistic issues involved in seventeenth century mathematics and science. We have all been thoroughly trained in algebra and calculus and have come to rely on this language and grammar as a dominant form of mathematical representation. We inherently trust that these symbolic manipulations will give results that are compatible with geometry; a trust that did not fully emerge in mathematics until the early works of Euler more than a century after Descartes. Such trust became possible because of an extensive set of representational experiments conducted throughout the seventeenth century which tested the ability of symbolic algebraic language to represent geometry faithfully **[5, 7]**. Descartes' *Geometry* is one of the earliest and most notable of these linguistic experiments. Because of our cultural trust in the reliability of symbolic languages applied to geometry, many of those schooled in mathematics today have learned comparatively little about geometry in its own right.

Descartes wrote for an audience with opposite predispositions. He assumed that his readers were thoroughly acquainted with geometry, in particular the works of Apollonius (ca. 200 BC) on conic sections **[1, 15]**. In order to appreciate the accomplishments of Descartes one must be able to check back and forth between representations and see that the results of symbolic algebraic manipulations are consistent with independently established geometrical results. The seventeenth century witnessed an increasingly subtle and persuasive series of such linguistic experiments in the work of Roberval, Cavalieri, Pascal, Wallis, and Newton **[8, 9]**. These led eventually to Leibniz's creation of a general symbolic language capable of fully representing all known geometry of his day, that being his "calculus" **[5, 7]**.

Because many of the most simple and beautiful results of Apollonius are scarcely known to modern mathematicians, it can be difficult to recreate one essential element of the linguistic achievements of Descartes—checking algebraic manipulations against independently established geometrical results. In this article I will ask the reader to become a kind of intellectual Merlin and live history backwards. After we explore one of Descartes' curve-drawing devices, we will use the resulting bridge between geometry and algebra to regain a compelling result from Apollonius concerning hyperbolic tangents. The reader can choose to regard the investigation either as a philosophical demonstration of the consistency between algebra and geometry or as a simple analytical demonstration of a powerful ancient result of Apollonius. By adopt-

[2] Descartes' contemporary, Fermat, did begin graphing equations but his work did not have nearly the philosophical or scientific impact of Descartes'. Fermat's original problematic contexts came from financial work rather than engineering and mechanics.

ing *both* views one gains a fully flexible cognitive feedback loop of the sort that my students and I have found most enlightening **[6]**.

I was recently discussing my work on curve-drawing devices and their possible educational implications with a friend. His initial reaction was surprise: "Surely you don't advocate the revival of geometrical methods; progress in mathematics has been made only to the extent to which geometry has been eliminated." This claim has historical validity, especially since the eighteenth century, but my response was that such progress was possible only after mathematicians had achieved a basic faith in the ability of algebraic language to represent and model geometry accurately. I argued that one cannot appreciate the profundity of calculus unless one is aware of the issue of coordination of independent representations. Many students seem to learn and even master the manipulations of calculus without ever having questioned or tested the language's ability to model geometry precisely. Even Leibniz, no lover of geometry, would feel that such a student had missed the main point of his symbolic achievement **[5]**. On this point my friend and I agreed.

Descartes' curve-drawing devices poignantly raise the issue of technology and its relation to mathematical investigation. During the seventeenth century there was a distinct turning away from the classical Greek orientation that had been popular during the Renaissance in favor of pragmatic and stoic Roman philosophy. During much of the seventeenth century a class in "Geometry" would concern itself mainly with the design of fortifications, siege engines, canals, water systems, and hoisting devices—what we would call civil and mechanical engineering. Descartes' *Geometry* was not about static constructions and axiomatic proofs, but concerned itself instead with mechanical motions and their possible representation by algebraic equations. Classical problems were addressed, but they were all transformed into locus problems, through the use of a wide variety of motions and devices that went far beyond the classical restriction to straight-edge and compass. Descartes sought to build a geometry that included all curves whose construction he considered "clear and distinct" **[11, 20]**. An examination of his work shows that what he meant by this was any curve that could be drawn with a "linkage," i.e., a device made of hinged rigid rods. Descartes' work indicates that he was well aware that this class of curves is exactly the class of all algebraic curves, although he gave no formal proof of this. This theorem is scarcely known among modern mathematicians, although it can be proved straightforwardly by looking at linkages that add, subtract, multiply, divide, and generate integer powers **[3]**. Descartes' linkage for generating any integer power was used repeatedly in the *Geometry* and has many interesting possibilities **[10]**.

This transformation of geometry from classical static constructions to problems involving motions and their resultant loci has once again raised itself in light of modern computer technology, specifically the advent of dynamic geometry software such as *Cabri* and *Geometer's Sketchpad*. Many new educational and research possibilities have emerged recently in response to these technological developments **[26]**. It seems, indeed, that seventeenth century mechanical geometry may yet rise from the ashes of history and regain a new electronic life in our mathematics classrooms. (It has always had a life in our schools of engineering, where the finding of equations that model motion has always been a fundamental concern.) My own explorations of seventeenth century dynamic geometry have been conducted with a combination of physical models and devices along with computer animations made using *Geometer's Sketchpad* **[18]**. The first figure in this paper is taken directly from Descartes, but all the others were made using *Geometer's Sketchpad*. This software allows a more authentic historical exploration since curves are generated from geometrical actions rather than as the graphs of equations. Static figures cannot vividly convey the sense of motion that is necessary for a complete understanding of these devices.[3] In the generation of the figures in this paper no equations were typed into the computer.

FIGURE 1 is reproduced from the (original) 1637 edition of Descartes' *Geometry* [**11**, p. 50]. Descartes described the device as follows:

> *Suppose the curve EC to be described by the intersection of the ruler GL and the rectilinear plane figure NKL, whose side KN is produced indefinitely in the direction of C, and which, being moved in the same plane in such a way that its diameter KL always coincides with some part of the line BA (produced in both directions), imparts to the ruler GL a rotary motion about G (the ruler being hinged to the figure NKL at L). If I wish to find out to what class this curve belongs, I choose a straight line, as AB, to which to refer all its points, and on AB I choose a point A at which to begin the investigation. I say "choose this and that," because we are free to choose what we will, for, while it is necessary to use care in the choice, in order to make the equation as short and simple as possible, yet no matter what line I should take instead of AB the curve would always prove to be of the same class, a fact easily demonstrated.*

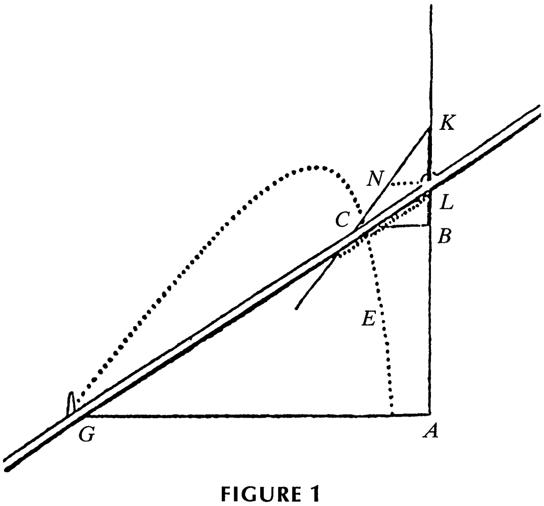

FIGURE 1
Descartes' Hyperbolic Device

Descartes addressed here several of his main points concerning the relations between geometrical actions and their symbolic representations. His "classes of curves" refer to the use of algebraic degrees to create a taxonomy of curves. He is asserting that the algebraic degree of an equation representing a curve is independent of how one chooses to impose a coordinate system. Scale, starting point, and even the angle between axes will not change the degree of the equation, although this "fact easily demonstrated" is never given anything like a formal proof in the *Geometry*. Descartes also mentioned here the issue of a judicious choice of coordinates, an important scientific issue that goes largely unaddressed in modern mathematics curricula until an advanced level, at which point geometry is scarcely mentioned.

Descartes went on to find the equation of the curve in FIGURE 1 as follows. Introduce the variables (Descartes used the term "unknown and indeterminate

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[3] Animated figures made in *Geometer's Sketchpad* are available from the author by e-mail (dennis@math.utep.edu).

quantities") $AB = y$, $BC = x$ (in modern notation, $C = (x, y)$), and then the constants ("known quantities") $GA = a$, $KL = b$, and $NL = c$. Descartes routinely used the lower case letters x, y, and z as variables, and a, b, and c as constants; our modern convention stems from his usage. Descartes, however, had no convention about which variable was used horizontally, or in which direction (right or left) a variable was measured (here, x is measured to the left). There was, in general, no demand that x and y be measured at right angles to each other. The variables were tailored to the geometric situation. There was a very hesitant use of negative values (often called "false roots"), and in most geometric situations they were avoided.

Continuing with the derivation, since the triangles KLN and KBC are similar, we have $\frac{c}{b} = \frac{x}{BK}$, hence $BK = \frac{b}{c}x$, hence $BL = \frac{b}{c}x - b$. From this it follows that $AL = y + BL = y + \frac{b}{c}x - b$. Since triangles LBC and LAG are similar, we have $\frac{BC}{BL} = \frac{AG}{AL}$. This implies the following chain of equations:

$$\frac{x}{\frac{b}{c}x - b} = \frac{a}{y + \frac{b}{c}x - b} \Leftrightarrow x\left(y + \frac{b}{c}x - b\right) = a\left(\frac{b}{c}x - b\right)$$

$$\Leftrightarrow xy + \frac{b}{c}x^2 - bx = \frac{ab}{c}x - ab$$

$$\Leftrightarrow x^2 = cx - \frac{c}{b}xy + ax - ac. \tag{1}$$

Descartes left the equation in this form because he wished to emphasize its second degree. He concluded that the curve is a hyperbola. How does this follow? As we said before Descartes assumed that his readers were well acquainted with Apollonius. We will return to this issue shortly.

If one continues to let the triangle NLK rise along the vertical line, and keeps tracing the locus of the intersection of GL with NK, the lines will eventually become parallel (see FIGURE 2), and after that the other branch of the hyperbola will appear (see FIGURE 3).

These figures were made with *Geometer's Sketchpad*, although I have altered slightly the values of the constants a, b, and c from those in FIGURE 1. In FIGURE 2, the line KN is in the asymptotic position, i.e., parallel to GL. I will hereafter refer to this particular position of the point K, as point O. In this position triangles NLK and GAL are similar, so $AK = AO = \frac{ab}{c} + b$ (the y-intercept of the asymptote). The slope of the asymptote is the same as the fixed slope of KN, i.e., b/c. (Recall that $KL = b$, $NL = c$, and $GA = a$.)

To rewrite Equation 1 using A as the origin in the usual modern sense, with x measured positively to the right, we can substitute $-x$ for x. With this substitution, solving Equation 1 for y yields

$$y = ab\frac{1}{x} + \frac{b}{c}x + \left(\frac{ab}{c} + b\right). \tag{2}$$

In Equation 2, the linear equation of the asymptote appears as the last two terms. In FIGURE 3, I have shown, to the right, the lengths that represent the values of the three terms in Equation 2, for the point P. (The labels 1, 2, and 3 represent, respectively, the inverse term, the linear term, and the constant term.) Term 3 accounts for the rise from the x-axis to the level of point O (the intercept of the asymptote). Adding term 2 raises one to the level of the asymptote, and term 1 completes the ordinate to the curve.

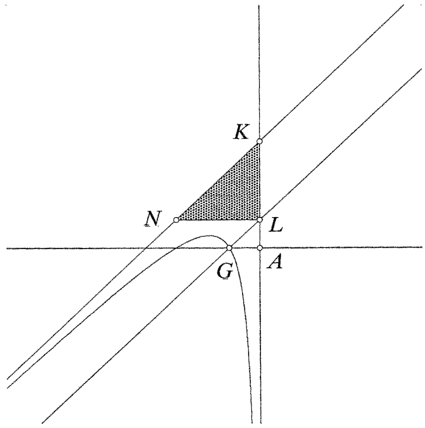

FIGURE 2
Descartes' Device in the Asymptotic Position

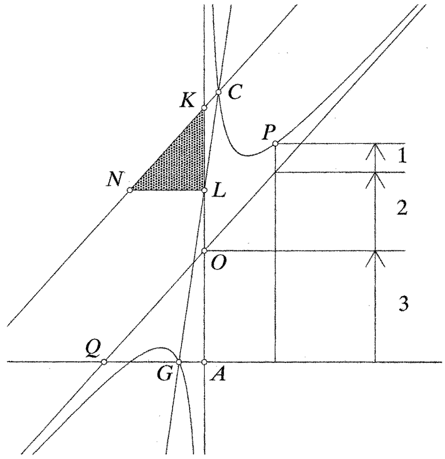

FIGURE 3
Geometrical Display of the Terms in the Hyperbolic Equation

As a geometric construction, the hyperbola is drawn from parameters that specify the angle between the asymptotes ($\angle NKL$), and a point on the curve (G). If one changes the position of the point N without changing the angle $\angle NKL$, the curve is unaffected, as in FIGURE 4. The derivation of the equation depends only on similarity, and not on having perpendicular coordinates. As long as GA (which determines the coordinate system) is parallel to NL, the derivation of the equation is the same except for the values of the constants $NL = c$, and $GA = a$ (both have become larger in FIGURE 4). Of course this equation is in the oblique coordinate system of the lines GA (x-axis) and AK (y-axis). It is the same curve geometrically, with the same form of equation, but with new constant values that refer to an oblique coordinate system. As long as angle $\angle NKL$ remains the same, and G is taken at the same distance from the line KL, the device will draw the same curve. This form of a hyperbolic equation, as an inverse term plus linear terms, depends only on using at least one of the asymptotes as an axis.

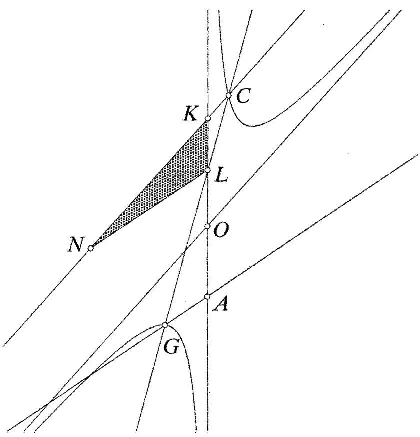

FIGURE 4
Hyperbola in Skewed Coordinates

I have encountered many students who are well acquainted with the function $y = \frac{1}{x}$, and yet have no idea that its graph is an hyperbola. Descartes' construction can be adjusted to draw right hyperbolas. Consider the special case in which the line KN is parallel to the x-axis (see FIGURE 5). The point G is on the negative x-axis. Let $KC = x$, and $AK = y$ (i.e., $C = (x, y)$), $AG = a$, and $KL = b$. Now $AL = y - b$, and since triangles LKC, and LAG are similar, we have $\frac{KC}{KL} = \frac{AG}{AL}$, or, equivalently, $\frac{x}{b} = \frac{a}{y-b}$. Hence the curve has equation

$$y = ab\frac{1}{x} + b. \tag{3}$$

A vertical translation by b would move the origin to the point O, and letting $a = b = 1$, would put G at the vertex $(-1, -1)$, yielding the curve with equation $y = 1/x$.

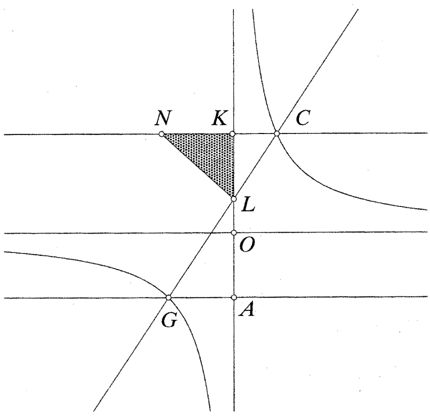

FIGURE 5
Device Adjusted to Draw Right Hyperbolas

Equation 3 can be seen as a special case of Equation 2, obtained by substituting ∞ for c, where c is thought of as the horizontal distance from L to the line KN. All translations and rescalings of the multiplicative inverse function can be directly seen as special members of the family of hyperbolas, using this construction.

Appollonius Regained

How do we know that these curves are, in fact, hyperbolas? Descartes said that this is implied by Equation 1. In his commentaries on Descartes, van Schooten gives us more detail [**11**, p. 55, note 86]. Once again these mathematicians assumed that their readers were familiar with a variety of ratio properties from Book 2 of the *Conics* of Apollonius [**1**, **15**] that are equivalent to Equation 1. I will not give a full set of formal proofs, but will instead suggest means for exploring these relations.

Several beautiful theorems of Apollonius concerning the relations between tangents and asymptotes are easily explored in this setting. Using the asymptotes of the curve in FIGURE 5 as edges to define rectangles, one sees that the points on the curve define a family of rectangles, all with the same area (see FIGURE 6). Indeed, if M and N are any two points on the curve, Equation 3 implies that $OPMS$ and $OQNR$ both have area equal to $a \cdot b$, the product of the constants used in drawing the curve. Another interesting geometric property is that the triangles TSM and NQU are always congruent. This congruence provides one way to dissect and compare these rectangles in a geometric manner [**17**].

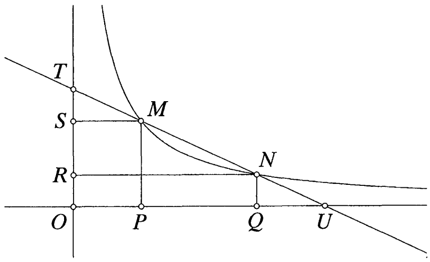

FIGURE 6
Hyperbola as a Family of Equal Area Rectangles

Approaching these equations analytically, assume that the curve in FIGURE 6 has the equation $x \cdot y = k$ (using O as the origin). Let $M = (m, k/m)$ and $N = (n, k/m)$, i.e., $OP = m$ and $OQ = n$. The line through M and N has equation $y = \frac{-k}{mn}x + \left(\frac{k}{m} + \frac{k}{n}\right)$. Hence $TO = \frac{k}{m} + \frac{k}{n}$, and, since $SO = \frac{k}{m}$, this implies that $TS = \frac{k}{n} = NQ$. Since triangles TSM and NQU are clearly similar, $TS = NQ$ implies that they are congruent and that $TM = NU$. Now let the points M and N get close to each other; then the line MN gets close to a tangent line, and one can perceive a theorem of Apollonius:

> *Given any tangent line to a hyperbola, the segment of the tangent contained between the two asymptotes is always bisected by the point of tangency to the curve* [**1**, **15**].

This property is a defining characteristic of hyperbolas. This simple and beautiful theorem immediately implies, among other things, that the derivative of $\frac{1}{x}$ is $\frac{-1}{x^2}$. (Look at the congruent triangles and compute the rise over run for the tangent.) This gives a student an independent geometrical check on the validity of the calculus derivation.

This bisection property of hyperbolic tangents is not restricted to the right hyperbola. Looking back at FIGURE 3 and Equation 2, one sees that any hyperbola coordinatized along both its asymptotes will always have an equation of the form $x \cdot y = k$ for some constant k. To see this, subtract off the linear and constant terms from the y-coordinate, and then rescale the x-coordinates by a constant factor that projects them in the asymptotic direction (in FIGURE 7 the new x-coordinate in this skew system is OQ). In the general case the curve can be seen as the set of corners of a family of equi-angular parallelograms, all with the same area. In FIGURE 7, for any two points M and N on the curve, the parallelograms $OQNR$ and $OPMS$ have equal

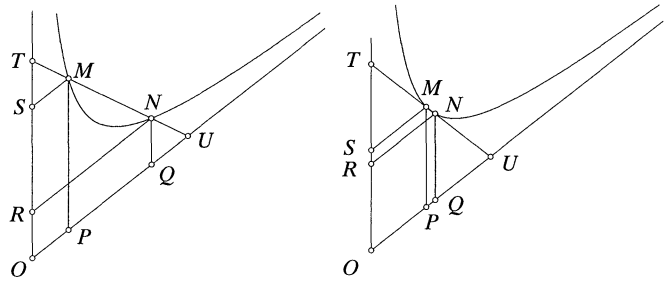

FIGURE 7
Bisection Property of Hyperbolic Tangents

areas. Since the triangles TSM and NQU are congruent, by letting M and N get close together one sees that any tangent segment TU is bisected by the point of tangency (M or N).

An alternative view of the situations just described is to imagine any line parallel to TU meeting the asymptotes and the curve in corresponding points T', M', and U'. Then the product $T'M' \cdot M'U' = TM \cdot MU$. That is to say, parallel chords between the asymptotes of a hyperbola are divided by the curve into pieces with a constant product. This follows from our discussion, because the pieces are constant projections of the sides of the parallelograms just discussed. This form of the statement was most often used by van Schooten, Newton, Euler and others in the seventeenth century. This statement (from Book 2 of Apollonius [**1**, **15**]) was traditionally used as an identifying property of hyperbolas. This constant product was given as a proof by van Schooten that the curve drawn by Descartes' device was indeed a hyperbola [**11**, p. 55]. Apollonius derived these properties directly from sections of a general cone.

In this way it is possible to investigate hyperbolas, using both geometric and algebraic representations, to create a complete cognitive feedback loop. Neither representation is used as a foundation for proof; instead, one is led to a belief in a relative consistency between certain aspects of geometry and algebra through checking back and forth between alternative representations. A calculus derivation of the derivative of $y = 1/x$ becomes, in this setting, a limited special case of the bisection property of hyperbolic tangents. It can be very satisfying to see symbolic algebra arrange itself into answers that are consistent with physical and geometric experience. Students of calculus can then experience the elation of Leibniz, as they build up a vocabulary of viable notation, capable of being checked against independently verifiable physical and geometric experience. Mathematical language is then seen as a powerful code for aspects of experience, rather than as the sole dictator of truth.

Conchoids Generalized from Hyperbolas

The hyperbolic device is only the beginning of what appears in Descartes' *Geometry*. He discussed several cases where curve-drawing constructions can be progressively iterated to produce curves of higher and higher algebraic degree [**11**, **10**]. It is usually mentioned in histories of mathematics that Descartes was the first to classify curves according to the algebraic degree of their equations. This is not quite accurate. Descartes classified curves according to *pairs* of algebraic degrees; i.e., lines and conics form his first class (he used the term *genre*), curves with third or fourth degree equations form his second class, etc. [**11**, p. 48]. This classification is quite natural if one is working with mechanical linkages and loci. With most examples of iterated linkage, each iteration raises the degree of the curve's equation by two, with some special cases that collapse back to an odd algebraic degree [**7**].[4] What follows is an example of such an interation based on the hyperbolic device.

[4] Descartes' linkages led directly to Newton's universal method for drawing conics, which is essentially a projective method [**7**, **23**]. This same classification by pairs of degrees is used in modern topology in the definition of "genus." The "genus" of a non-singular algebraic plane curve can be thought of topologically as the number of "handles" on the curve when defined in complex projective space. In complex projective space, linear and quadratic non-singular curves have genus 0, and are topologically sphere-like. Similarly, curves of degrees 3 and 4 are topologically torus-like, and have genus 1. Curves of degrees 5 and 6 are topologically double-holed and have genus 2, etc. In the real model, (i.e., when considering only real solutions of one real equation in 2 variables) the genus 0 curves consist of at most one oval when you join up the asymptotes. The genus 1 curves will have two ovals, which is what you'd expect when cutting through a toric by a plane, etc. (This comment was made to me by Paul Pedersen.)

Descartes generalized the previous hyperbola construction method by replacing the triangle KLN with any previously constructed curve. For example, let a circle with center L be moved along one axis and let the points C and C' be the intersections of the circle with the line LG, where G is any fixed point in the plane and LG is a ruler hinged at point L just as in the hyperbolic device (see FIGURE 8). Then C traces out a curve of degree four, known in ancient times as a *conchoid* [**11**, p. 55]. The two geometric parameters involved in the device are the radius of the circle (r), and the distance (a) between the point G and the axis along which L moves.

FIGURE 8 shows three examples of conchoids for $a > r$, $a = r$, and $a < r$. If the curve is coordinatized along the path of L, and a perpendicular line through G (OG), then its equation can be found by looking at the similar triangles GOL and CXL (top of FIGURE 8). Since $GO = a$, $LC = r$, $CX = y$, $OX = x$, and $XL = \sqrt{r^2 - y^2}$, one obtains the ratios of the legs of the triangles as follows: $\dfrac{\sqrt{r^2 - y^2}}{y} = \dfrac{\sqrt{r^2 - y^2} + x}{a}$. This is equivalent to $x^2y^2 = (r^2 - y^2)(a - y)^2$, an equation of fourth degree, or of Descartes' second class. (The squared form of the equation has both branches of the curve, above and below the axis, as solutions.)

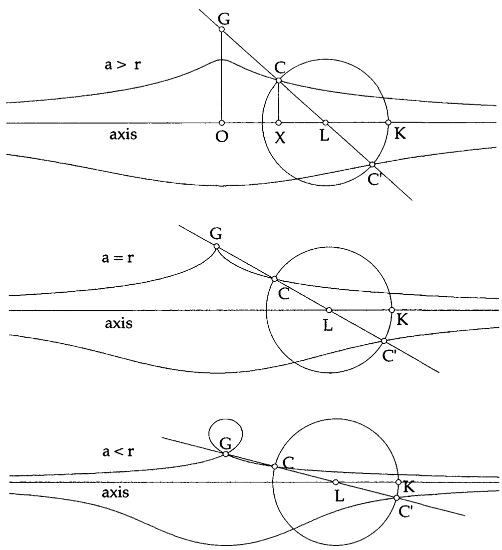

FIGURE 8
Conchoids Drawn by Dragging a Circle Along a Line

This example demonstrates Descartes' claim that, as one uses previously constructed curves to draw new curves, one gets chains of constructed curves that go up by pairs of algebraic degrees. Descartes called the conchoid a curve of the second class, i.e., of degree three or four. Dragging any rigid conic-sectioned shape along the axis, and drawing a curve in this manner will produce curves in the second class. Dragging curves of the second class will produce curves of the third class (i.e., degree five or six), etc. Descartes demonstrated this general principle through many examples [**11**, **7**, **10**], but he offered nothing like a formal proof, either geometric or algebraic. His definition of curve classes was justified by his geometric experience.

Notice that when $a \leq r$, the point G becomes a cusp or a crossover point. When singularities like cusps or crossover points occur, these tend to occur at important parts of the apparatus, like a pivot point (such as G) or a point on an axis of motion.

Other important examples of this phenomena can be found in Newton's notebooks [**22**, **23**]. I am not asserting any particular or explicit mathematical theorem here. This general observation is based upon my own historical research and empirical experience with curve-drawing devices. There are probably several ways to make this observation into an explicit mathematical statement, subject to proof (Newton attempted several [**23**]). There are many open questions concerning these forms of curve iteration and the relations between the parts of the physical devices and the singularities of the curves [**7**]. Students might benefit from such empirical experience —regardless of the extent to which they eventually formalize that experience in strictly algebraic or logical language. An instinctual sense of where curve singularities might occur is fundamentally useful in many sciences [**2**]. Modern computer software makes such investigations routinely possible with a minimum of technical expertise.

Conclusion

Descartes wrote his *Rules for the Direction of the Mind* [**12**] in 1625, twelve years before he would publish his famous *Geometry*. In this earlier work he emphasized the importance of making strong connections between physical actions and their possible representations in diagrams and language. Here are a few quotes:

> *Rule 13: If we understand a problem perfectly, it should be considered apart from all superfluous concepts, reduced to its simplest form, and divided by enumeration into the smallest possible parts.*
>
> *Rule 14: The same problem should be understood as relating to the actual extension of bodies and at the same time should be completely represented by diagrams to the imagination, for thus will it be much more distinctly perceived by the intellect.*
>
> *Rule 15: It is usually helpful, also, to draw these diagrams and observe them through the external senses, so that by this means our thought can more easily remain attentive.*

These lines from Descartes sound much like parts of the hands-on, problem-solving educational philosophy of mathematics put forth by the National Council of Teachers of Mathematics [**21**]. Descartes' entire approach to mathematics had problem solving as its foundation [**14**], but we must not allow ourselves to read into him too modern a perspective. He was constructing a new method of mathematical representation that responded to both the new symbolic language of his time (algebra) and to the new technology of his time (mechanical engineering). He was not seeking the broad educational goals of the NCTM. In fact, his *Geometry* was not widely read in the seventeenth century until it was republished, in 1657, with extensive commentaries by Franz van Schooten.

Nonetheless, Descartes' approach to geometry through curve-drawing devices and locus problems has important implications for education. His work connects important classical and Arabic traditions with modern algebraic formalisms [**7**]. It provides the missing linkages (pun intended). These linkage and loci problems, combined with the new dynamic geometry software, allow a new kind of exploration of curves that could go far towards ending the isolation of geometry in our mathematics curriculum. One can use geometrical curve generation to recreate calculus concepts such as tangents and areas in a much more elementary and physical setting [**7**, **8**, **10**], as well as to explore complicated questions about algebraic curves left open since the seventeenth century [**7**, **23**]. Computer graphic techniques have already led to new branches of mathematics, such as fractals. Perhaps a new phase of computer-assisted empirical geometrical investigation of curves and surfaces has already begun. If this new beginning proves as revolutionary as the century that began with Descartes' *Geometry*, then we are in for some very exciting times.

Acknowledgment. This research was funded by a grant from the National Science Foundation.

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