6.3. Angles

In this section, we explore two different ideas of angles: as an amount of rotation, and as the amount of opening between two rays.

The next subsection addresses operations with angles: comparing the sizes, adding, subtracting and subdividing/splitting angles.

The idea of adding angles leads to an exploration of (interior) angles in polygons.

Finally, the idea of angle as a turn leads to an introduction to the computer language Logo, which allows you to control a "turtle" on the screen that traces shapes by dragging its tail as it walks.

A. Two concepts of angle

What is an angle? Most people will talk about the space between two intersecting lines, or something similar. While this is an important aspect of the idea of angle, the idea that comes most easily to young children is related to moving their bodies: rotation.

The rotation idea of of angle: An *angle* is an amount of turning, or the amount of rotation.

One unit for measuring angles is a *turn*, as described below.

Class activity 1. Turns.

a) Face the front of the classroom. Do a full turn (all the way around). Which way are you facing?

b) Again facing front, rotate to your right a half turn. Which way are you facing?

c) Again facing front, rotate to your right a quarter turn. Which way are you facing? If you did a total of 4 quarter turns, that would be a full turn. A quarter turn is also called a right angle.

d) A third of a turn is a little harder to judge. Try doing a third of a turn. Repeat a total of 3 times, and you should be facing your original direction.

Angles as turns happen all the time in daily life: when you're walking, driving, or riding a bike. Sometimes the turning is just turning, as when you're spinning in place. More often, you travel straight for a while, then make a turn, then travel straight some more. Frequently the traveling and turning happen at the same time, as when you're driving on a curved road.

The "opening between two rays" idea of angle: An *angle* is the amount of opening between two rays that have a common endpoint. (Recall that a ray is a half line, which goes indefinitely in one direction, and has an endpoint in the other direction.) The common endpoint of the two rays is called the *vertex* (corner) of the angle.

In three dimensions, a *dihedral angle* is the amount of opening between two half-planes. (A half plane is the part of a plane on one side of a line in the plane.)

Sometimes it matters which direction in which an angle is measured (clockwise or counter-clockwise); these are called *directed angles.* In the figures below, there are four possible ways to interpret the angle formed by rays L and ray M.

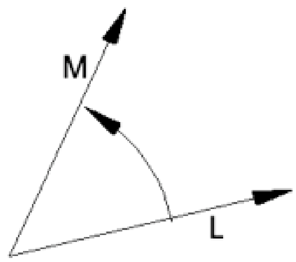

Figure 1. Angle from ray L to ray M, counterclockwise.

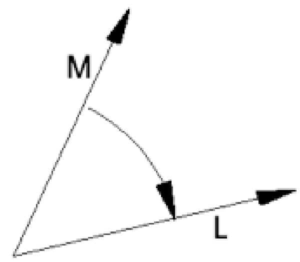

Figure 2. Angle from ray M to ray L, clockwise.

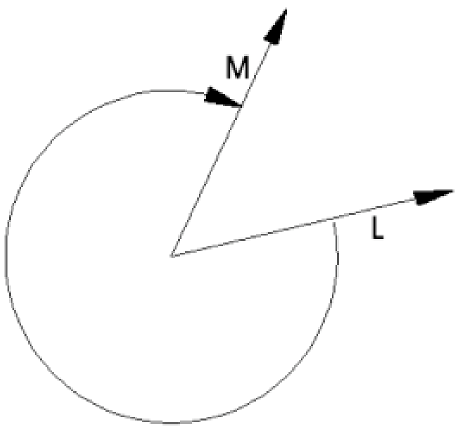

Figure 3. Angle from ray L to ray M, clockwise.

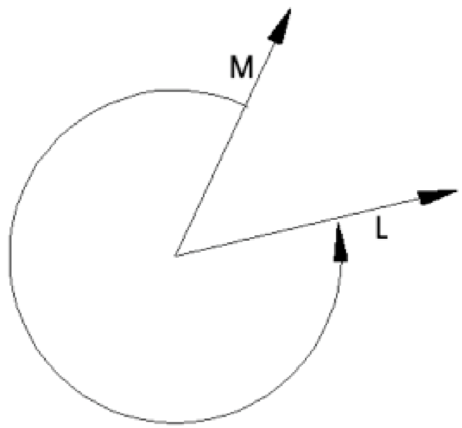

Figure 4. Angle from ray M to ray L, counterclockwise.

Figures 1 and 2 are like opening a book a little bit, just enough to peek inside. Figures 3 and 4 like opening it too far. (The librarian is rushing over, saying, "Stop! You'll break the spine!")

If rays L and M represent the starting and ending directions of a rotation, there are even more possible measures for the angle. Two are shown below.

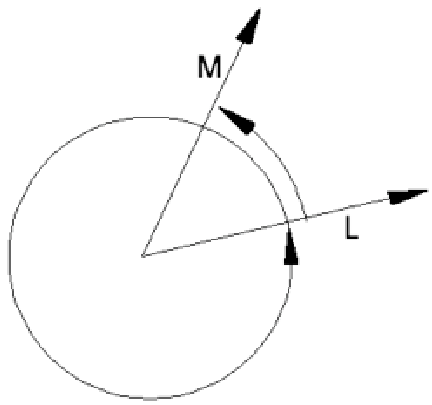

Figure 5. A rotation from ray L to ray M that has gone around a full turn and a little more.

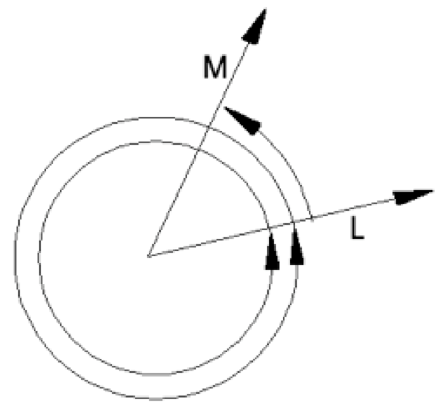

Figure 6. A rotation from ray L to ray M that has gone around two full turns and a little more.

Another variation on the "two rays" idea of an angle the "wedge" idea: imagine them as being drawn on paper, then cut along the rays. You will get two angles, one with a sharp point, and the other with a narrow gap of missing paper.

The big wedge (opening the book too far) is called a *reflex angle.* When you see an angle formed by two rays in a textbook, they usually mean the smaller wedge. If not, the angle to look at is shown with an arc, as in Figures 1-6.

Class activity 2. Use an angle on paper to determine a turn. Draw an angle formed by two rays on a piece of paper. Put the paper on the ground. Stand at the vertex of the angle, face in the direction of one of the rays, then rotate to the direction of the other ray. There are many ways to do this. Try starting with one ray, then the other. Try rotating to your right, or to your left. Try rotating several turns before ending in the final direction.

B. Units for measuring angles

Degrees. Turns are one unit for measuring angles. Another common unit to measure angles is the degree, which was invented by the ancient Babylonians (in Iraq!) A full turn is 360 degrees, written 360°. Nobody is sure why they decided to split a full turn into 360 parts, but it might be for astronomical purposes. There are 365 nights in a year, and every night the stars appear to rotate a little bit. After a year, they are back to their original position. Another reason is that 360 has a lot of factors; the fractions 1/2 of a turn, 1/3 of a turn, 1/4 of a turn, and many others, are a whole number of degrees.[1]

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[1] Another unit for measuring angles is in *radians*, which measures the distance along a circle with radius 1 length unit to measure an angle. A metric unit for angles that didn't catch on is the *grad*, which is 1/100 of a right angle.

Here is a ratio table that summarizes conversions between turns and degrees for some common angles.

Turns	1	2	3	1/2	1/4	1/8	1/16	1/3	1/6	1/12	1/24
Degrees	360	720	1080	180	90	45	22.5	120	60	30	15

A quarter turn is also called a right angle; a half turn is called a straight angle, because it looks like a straight line with a dot (the vertex) in the middle.

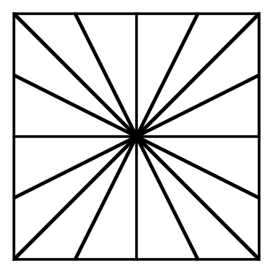
An *acute* angle is one that is less than a right angle. An *obtuse* angle is more than a right angle.

Class activity 3. Make two protractors with paper folding. A protractor is a device for measuring angles. Usually they are in the form of a half circle, so that you can measure angles up to a half turn, but there are also full circle protractors. You will make two protractors from paper, one of which is square.

Materials, Part a: Patty paper: squares of lightly waxed paper used to separate hamburger patties in restaurants. Buy it at a restaurant supply store or order onlilne from a math education supply company. Unlined white paper is OK, too.

Part b: A circle of plain paper, with the center marked, or compass and scissors for making your own.

Part c: A ready-made protractor



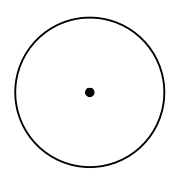
a) **First protractor:** Start with a square piece of paper. Fold it in half, matching opposite edges. Open it and fold in the other way, matching the two existing fold lines. Repeat, making more folds, until the folds are evenly spaced. Now do the folding one more time, matching previous fold lines.

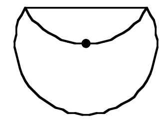
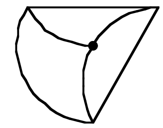
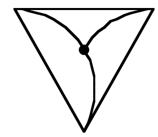
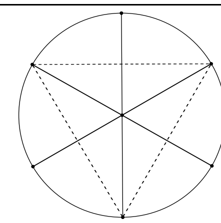
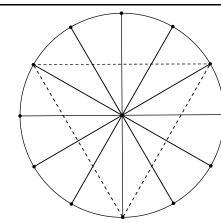
Pick one ray from the center as 0. Go around counterclockwise and mark the rays with the amount of turn from 0. Also mark each ray with number of degrees.

Using a ruler, draw some random angles on a piece of paper and use your homemade protractor to measure the angles. Measures will be approximate.

On this protractor, the smallest angles measure 22.5 degrees.

b) **Second protractor:**

Mark a point in the middle of a piece of paper. Use a compass to draw a fairly large circle with the point as its center. Cut it out.	

Make a fold so that the edge of the circle lies exactly on the center of the circle.	
Make another fold, starting at one of the ends of the existing fold, so that the edge of the circle lies exactly on the center of the circle.	
Make a final fold that goes between the two remaining ends of the folds, so that the edge of the circle lies exactly on the center of the circle. You should have a triangle (what kind?)	
Open the paper. Fold the circle in half so that the fold goes through one of the vertices of the triangle. Open, and repeat for the other two vertices.	
Open the paper. Fold the circle in half so that the fold bisects the smallest angle through the center (match two of the closest fold lines.) Open, and repeat so that the fold lines are equally spaced.	

Unfold the whole thing. Mark the rays starting with 0, both in turns and in degrees.

Measure your random angles again with the circle protractor. Are the numbers you get consistent with the previous measurements?

c) Put the purchased protractor on top of one of your homemade ones. Does it help explain what numbers are marked on the purchased protractor?

Think about how you would tell students to align a purchased protractor to measure angles. What part of the protractor goes on the vertex of the angle? Where does one ray of the angle go on the protractor? The other ray? Which number is the correct measure, and why are there two numbers for each mark?

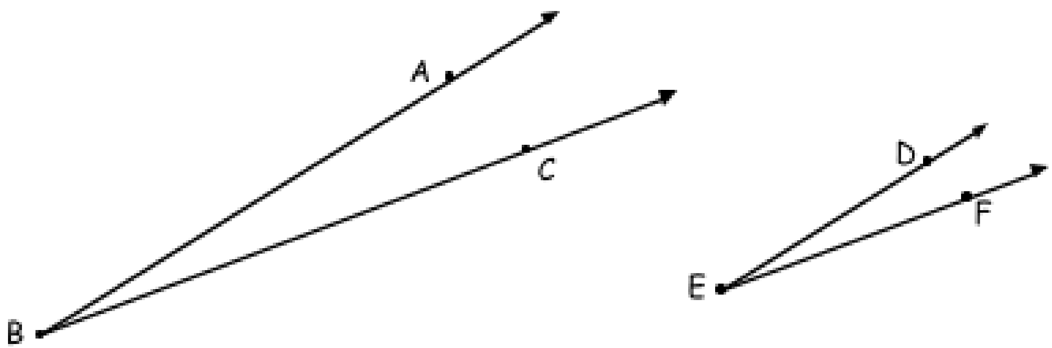
Measure your random angles again with the purchased protractor. How accurate do you think your measurement is? To the nearest degree? To the nearest 2 degrees? Does is depend on how sharp the pencil was that you used to draw the line?

Euclid, the Greek geometer who wrote the book on which our high school geometry is still based, used a right angle as the unit for measuring angles. Instead of saying that several angles added to 180 degrees, he said that the angles make two right angles.

C. Comparing, adding, and subtracting angles.

Etty Wanda says:

Today this kid in my class said that angle ABC is bigger than angle DEF. Mr. Groener got mad at him today instead of me!



Discussion question 1. The picture of angle ABC looks bigger in some way. In what way is the picture bigger? Is the angle actually bigger?

Comparing, adding, subtracting, and halving angles as wedges.

To compare angles if they are cut out as wedges is easy: put one on top of the other, matching the vertices and one side. Whichever one sticks out on the other side is bigger.

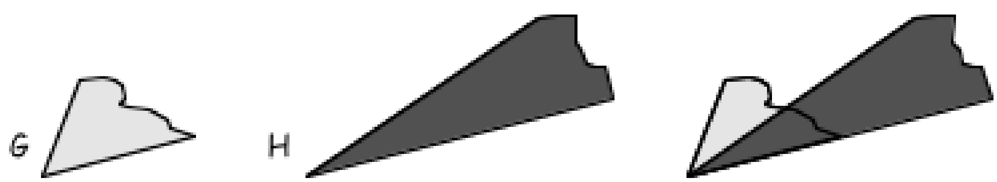


Figure 7. Angle G is bigger than angle H.

One way to concentrate on what's important in measuring angles is to make a piece of paper with a hole in the middle. Put the paper over the angles with the vertex showing through the hole. The parts of the angle(s) you can't see don't matter.

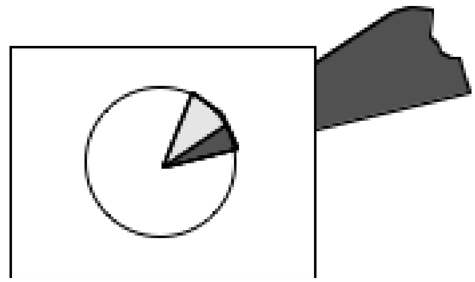


Figure 8. Pay attention to only what shows through the hole.

To add angles as wedges, put the vertices together, and align a ray of one with the ray of the other, side by side. The sum is the combined angle between the two outer rays.

Figure 9. Adding angle G and angle H.

To subtract angles as wedges, line them up on top of each other as to compare them. The difference (answer of subtraction) is the angle sticking out. You could think of this as the take away or the comparison idea of subtraction.

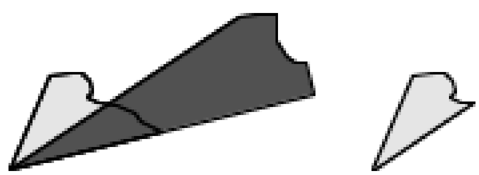


Figure 10. Subtracting angle H from angle G.

To divide an angle in half (bisect it), make a fold that matches the two rays and goes through the vertex.

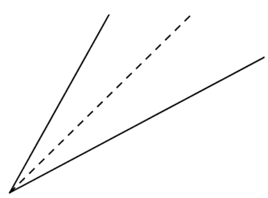
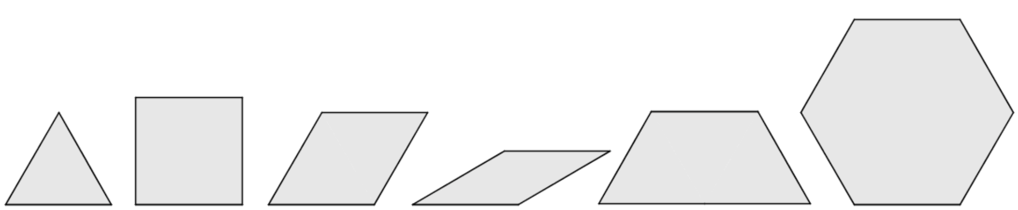


Figure 11. The dotted line bisects the angle formed by the solid rays.

Class activity 4. Physical angle addition and angles in pattern blocks

Materials: Pattern blocks, and a piece of paper with a hole in it, as in Figure 8



Using physical angle addition, find the measure of each angle in each of the 6 pattern blocks.

Example: the square. (We already know the angles of a square are right angles. We're using this known fact to illustrate the angle addition method.)

Put enough squares that one vertex of a square is completely surrounded. If necessary, use the paper to mask the other parts so that you can concentrate on this one vertex.

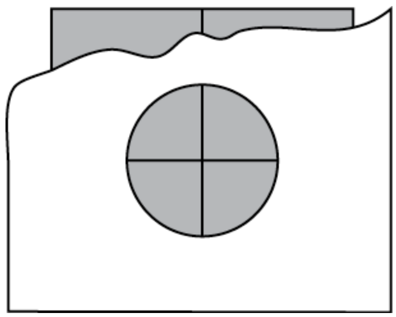


Figure 12. Four squares meet at each vertex.
The paper mask lets you concentrate on just that vertex.

The angles at the vertex make a full turn, or 360°. Because this total angle is made of four equal angles (each from a separate square), each angle is 90°. The four angles of a square are equal, so they all measure 90°.

Comparing, adding, and subtracting angles as rotations.

To add angles as rotations, do one rotation, and keep going, doing the second rotation. The sum is the total amount of rotation from your original direction.

To subtract angles as rotations, do one rotation, and keep going, and then do the second rotation in the reverse direction. (This is the "add the opposite" idea of subtraction, very similar to the way you subtract numbers on a number line.) A second way to think of subtracting angles is that the difference between the two angles is the net amount of rotation from your original direction to your final direction, or, in other words, the change in direction. This is the idea of subtraction as comparison, or as amount of change.

D. Angles in polygons

The next activity focuses on two ways of thinking of the angles in a polygon: one by walking to trace out the polygon, where the angles are turns, and one by looking at the opening between sides.

The rotation concept of angles is easier to understand with physical actions. To plan for a 90 degree turn to the right, point your left arm forward, and your right arm out to the side, so that your arms form a 90 degree angle. Make a note of which direction your right arm is pointing, then turn to face that direction.

Class activity 5. Interior and exterior angles in polygons.

Materials: A large loop of rope.

Find a big open space. Three people stand inside the loop of rope, holding it at waist level.

a) Interior angles in a triangle. Move so that the rope is stretched tight and forms a triangle. Each vertex person faces the inside and extends their arms in the directions of the rope, forming an angle. These are the *interior angles* of the triangle. Each person, with advice from the class, estimates the measure of their angle. Is it a right angle, or more or less than a right angle? Can you make a triangle with one right angle? With one obtuse angle? Two obtuse angles? Three acute angles?

b) Now a fourth person walks around the outside of the triangle. At each vertex, stop and estimate the angle of the turn: the change in direction from straight ahead to the direction along the next side. This is the *exterior angle* at that vertex of the triangle. At each vertex, use arms to judge whether the turn you are about to make is more or less than 90 degrees. Repeat as you go around the triangle. End at the starting point, facing the same starting direction.

c) How are the interior and exterior angles at each vertex related? At each vertex, have both the vertex person and the walker stand together, and show the angles with their arms.

d) How much of a total rotation did the walker do?

   Before the walker starts, make a note of which directions s/he was facing. Keep track of how many times s/he was facing that direction as s/he walks around the triangle, even if it was only for an instant. If the walker was only facing that direction at the start and end of the trip, then the total rotation for the trip is one turn.

   Have a walker stand in the middle of the triangle and pretend to walk around. Actually do the correct turns, but just walk in place. Note the total rotation for the trip.

e) Repeat with other volunteers, forming different triangles with different angles. Or have everyone walk around the triangle. Ask the same questions:

   Estimate the size of the interior angle at each turn.

   Estimate the size of the exterior angle at each turn.

   What is the total rotation (the sum of the exterior angles) for the trip?

f) Repeat with a quadrilateral. Start with a square or rectangle, but then make some quadrilaterals with other angles.

g) Repeat with 5-sided and 6-sided figures, and a circle.

What can you say about the sizes of the interior angles as you increase the number of sides?

What can you say about the total rotation as you increase the number of sides?

An *interior angle* at a vertex in a cut out polygon is the wedge at the vertex. In the rotation idea of angle, an interior angle is the angle you rotate as you turn from the vertex to your right to the vertex to your left, facing the inside of the polygon from a vertex. (Or you could rotate from left to right.)

An *exterior* angle at a vertex in a polygon is the angle between an extended side and the next side, measured outside the triangle. In the rotation idea of angle, an exterior angle is the angle you rotate as you walk around that corner of the polygon. It's the turn you make to keep from going straight when you get to the vertex. Note that the exterior is *not* the rest of the way around; that's the reflex angle.

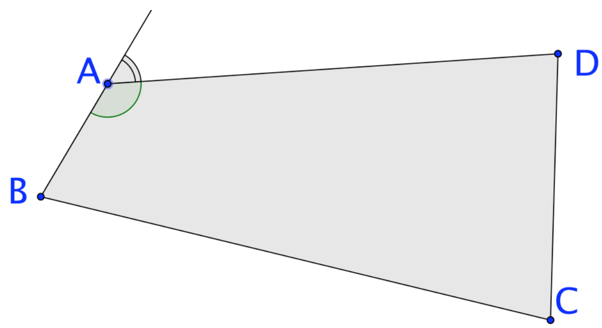


Figure 13. The interior and exterior angle at A.
The exterior angle is the turn you make when walking from B, around A, towards D.

Notice that an interior angle and its corresponding exterior angle add to a straight angle.

Terminology: Two angles are *supplementary* if they add to a straight angle, 180°. Often these angles share a side, physically making a straight angle. Supplementary can also refer to angles in different places that numerically add to 180°.

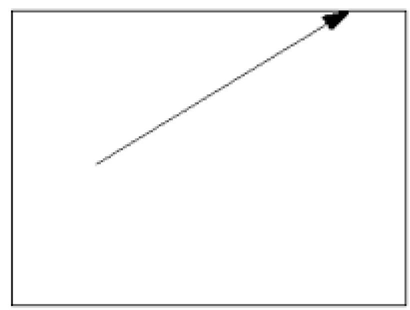
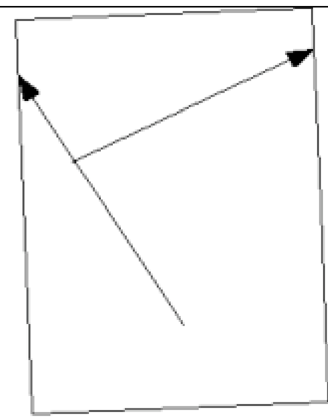
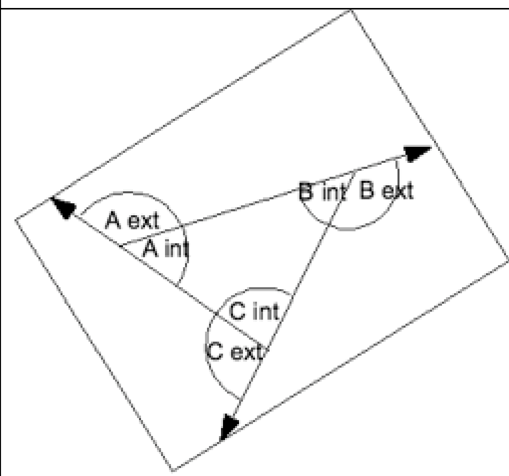
Two angles are *complementary* if they add to a right angle, 90°.

Class activity 6. Interior and exterior angles in polygons with paper.

Materials: Unlined paper, preferably in several colors; ruler, scissors, tape

a) Interior and exterior angles in a triangle.

On one color of paper, make a large triangle by drawing three rays, as follows.

Make a point on your paper, then draw a ray from the point off the edge of the paper.	
Rotate your paper. Draw a point on the first ray, then draw another ray from the new point off the edge of the paper.	
Rotate the paper again, in the same direction. Draw a point on the second ray. Draw a ray from that point that goes through the first point, then off the edge of the paper. Label the vertices of the triangle A, B, C, then mark both the interior and exterior angles at each vertex.	

Cut along only the rays. You should have a triangle and 3 outside pieces.

Add the exterior angles physically, by putting all the vertices together, and taping the pieces edge to edge.

What is the sum of the exterior angles of your triangle? Check whether other people got the same answer for different triangles.

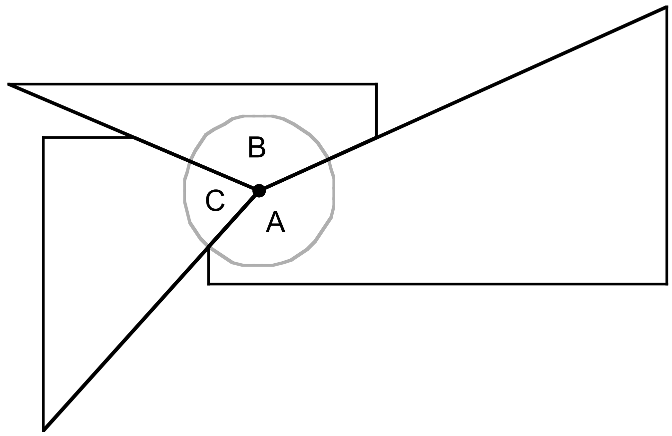


Figure 14. The sum of the exterior angles of a triangle.

Cut or tear the triangle into three pieces, with one angle on each piece. Add the interior angles physically.

What is the sum of the interior angles of your triangle? Check whether other people got the same answer for different triangles.

Tape the cutout to the black- or whiteboard so that everyone can see many examples.

b) On a new piece of paper in a different color, use 4 rays to construct a quadrilateral (polygon with 4 sides.) Repeat the cutting, and find the sum of the exterior and interior angles of your quadrilateral. Discuss results, and tape examples to the board.

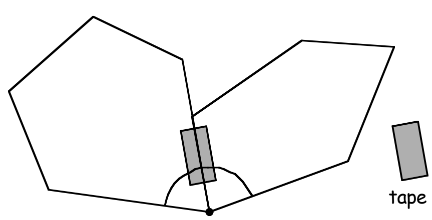


Figure 15. Two exterior angles of a pentagon.

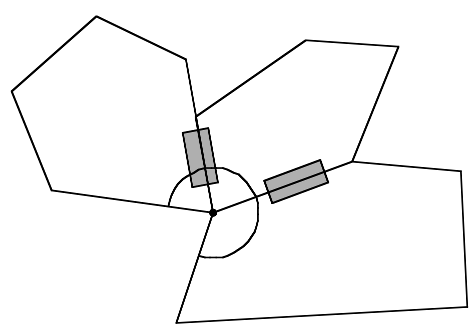


Figure 16. Three exterior angles of a pentagon.

c) On a new piece of paper in another color, use 5 rays to construct a pentagon (polygon with 5 sides). Repeat the cutting, and find the sum of the exterior and interior angles of your pentagon. For the interior angles, tape each new angle onto one edge of the previous angle, as shown. Keep going around like a ramp, forming a second layer, like a parking garage ramp, or a spiral staircase, if you need to.

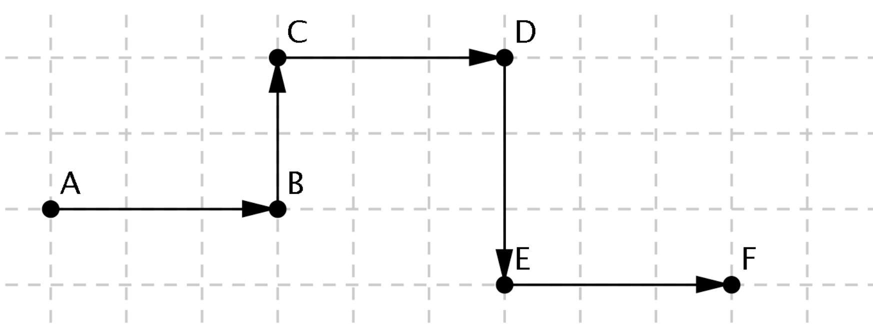
If you have more than one layer, make a careful cut so that you have a complete turn, and some extra, in two separate pieces. What is the total angle of rotation of your spiral staircase?

d) Fill in the table below, and make a conjecture what will happen for polygons with more sides.

Number of sides	Sum of exterior angles	Sum of interior angles
3		
4		
5		
6		
7		
8		
9		
10		

E. Angles in navigation vs. angles in geometry

Example 1. What is the total rotation for the trip shown below? You could also find the total distance, but that's not the question here. Try to imagine traveling along the path. For example, as you come to point B, you are pointing to the right side of the paper. At B, you turn towards the top of paper, which is a left turn. At C, there is a turn to the right, which cancels the turn to the left. That is, you are again facing the right side of the paper.



Vertex	amount of rotation, in turns	Comment	total rotation
B	-1/4	turn 90° left	-1/4
C	1/4	turn 90° right	0
D	1/4	turn 90° right	1/4
E	-1/4	turn 90° left	0

The total rotation of the trip is 0 turns; you did not turn completely around, and you ended up facing the same directions as you started.

Class activity 7. Logo and turtle graphics. The programming language Logo was designed in part to allow students to tell a computer to draw things. In this activity, we'll be the turtles. The idea is that you are a turtle dragging its tail in the sand, tracing out shapes. Commands include "forward X" (meaning walk forward X units), "right Y" and "left Z" (meaning turn right an angle of Y or left an angle of Z). In this activity, angles will be described by fractions of a turn, though in the program, angles are measured in degrees.

What shape is traced by these sets of instructions? Check by doing it with a partner on a sidewalk and having them trace your path with chalk.

Then check with a Logo program, using degrees instead of turns as the unit.

a) Right 1/4 turn, then forward 2 steps; repeat these two instructions until you get back to where you started.

b) Right 1/2 turn, forward 2 steps; repeat these two instructions.

c) Right 1/4 turn, then forward 2 steps; left 1/4 turn, then forward 2 steps; repeat these four instructions.

d) Right 1/8 turn, forward 4 steps;
right 3/8 turn, forward 2 steps;
right 1/8 turn, forward 4 steps;
right 3/8 turn, forward 2 steps.

See the end of the section for a summary of Logo commands.

F. The Logo computer language

There are several online versions of Logo that are easy to use, and you don't have to download any software. There are other versions you can download (see the class Moodle page for links.)

- http://home.alphalink.com.au/~rhduncan/logo/index.html

  (this is in Australia)

- http://www.mathsnet.net/logo/turtlelogo/

(same program as the previous, but in the U.K.)

- http://kaminari.scitec.kobe-u.ac.jp/java/logo/

  (This one is in Japan. There are buttons for commands so you don't have to type much. Commands are slightly different: "pen on" instead of pen down, loop ... end; loop instead of repeat [...]. Press "Paint" to draw the picture after entering the commands.

- http://www.psinvention.com/Turtle.htm

  (In this version of Logo, you can repeat a command by repeatedly pressing the "Add" button.)

Most of the programs have a separate line or window to type the commands.

Note that the unit of distance in this program is a pixel: the length or width of one dot of light on the screen. So you probably won't be able to see the effect of "forward 1". The programs' screens are on the order of 400 pixels by 400 pixels.

Warning: when you go off the right side, the turtle reappears on the left, and the same for top and bottom. If you get weird pictures, this may be what happened. Try the command "forward 400." What happens, and why?

Logo commands.

There is some variation in the abbreviations between different implementations.

Letters in *italics* are variables: replace the letter with a specific number. Hit the Return or Enter key at the end of each command. (Sometimes you need to press another key with Enter.)

Full command	**Abbrev-iation**	**What it does**	**Example**
draw		Opens or clears the drawing window	
	cl or cs	Clears the drawing window and puts the turtle in the center, facing up.	
home	h	Returns the turtle to its home position and direction, in the center of the screen and facing the top.	
forward *D* *D is a distance in pixels*	fd *D*	Turtle goes forward *D* units. If the pen is down, it traces its path.	fd 20

right *A* *A is an angle in degrees*	rt *A*	Turtle turns *A* degrees to its right, but otherwise doesn't move	rt 90
back *D*	bk *D*	Turtle goes backward *D* units. If the pen is down, it traces its path.	bk 20
penup	pu	Picks up the "pen" so the turtle can move without leaving a trace	
pendown	pd	Puts the "pen" down so the turtle can leave a trace	
repeat *N* [ ] *N is the number of times to repeat*		Repeats whatever is inside the square brackets *N* times. Include the brackets when typing the command.	repeat 4 [fd 20 rt 90]

Example 2. Constructing a regular pentagon. Number of sides/angles = K = 5.

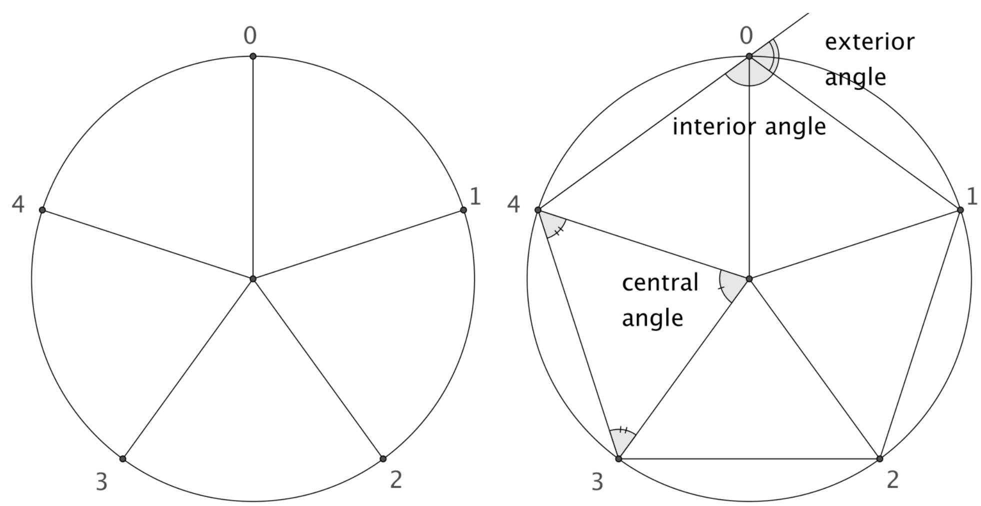
Materials: Unlined paper, compass, protractor, straightedge

Use a compass to draw a circle. Draw a radius.

Use a protractor to measure a central angle of 1/5 of a turn, which is 72 = 360/5 degrees.

Repeat until the circle is divided into 5 equal sectors, each with a 72 degree angle. Mark the points on the circle where the radii end.

Make a regular pentagon by connecting the points with line segments.



What is each interior angle on the pentagon? Figure it out by reasoning from what you know about angles and polygons.

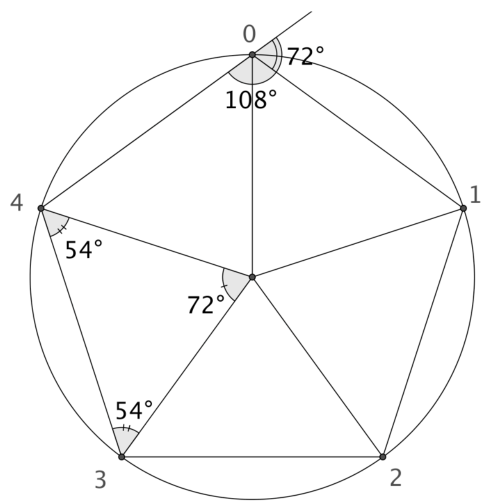
What is each exterior angle on the pentagon?

Method 1: The sum of interior angles on a pentagon is 540 (from the class activity), and to be a *regular* pentagon, there must be 5 equal angles. So each is 540/5 = 108 degrees.

The sum of the exterior angles is 360, so each exterior angle is 360/5 = 72 degrees.

Method 2: The sum of the exterior angles is 360, so each is 360/5 = 72 degrees. The exterior and interior angles at a vertex make a straight angle, so the interior angle is $180^\circ - 72^\circ = 108^\circ$

Method 3: The radii in the construction divide the pentagon into 5 isosceles triangles which are congruent to each other. So the 5 angles meeting at the center must add to 360°, making each angle 72 degrees. Since the angles in one triangle must add to 180°, so the other two angles must add to $180^\circ - 72^\circ = 108^\circ$. This is an isosceles triangle, so these two angles must each have half of 108°. But the interior angle in the pentagon is formed by two of these angles, so the interior angle in the pentagon is 108°.



The exterior angle is the turn you would make in walking around the pentagon, so a Logo program to trace the pentagon is

repeat 5 [forward 100 right 72]

The central angle and the exterior angle are both 72 degrees. Is this a coincidence, or will it be true for every regular polygon? Why?

Constructing a regular 5-pointed star

If you number the points starting with 0, then you can describe the sequence of points you connect using arithmetic. To make the regular pentagon, we added 1, to make the sequence 0, 1, 2, 3, 4. If, instead, you repeatedly add 2 (count on 2 points,) the points you get are 0, 2, 4, 1, 3, 0. The result is a 5-pointed star.

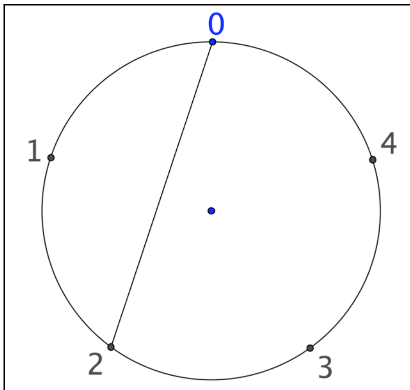


Figure 17. Segment from 0 to 2.

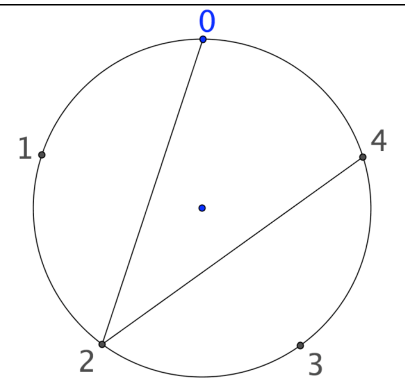


Figure 18. Segment from 2 to 4.

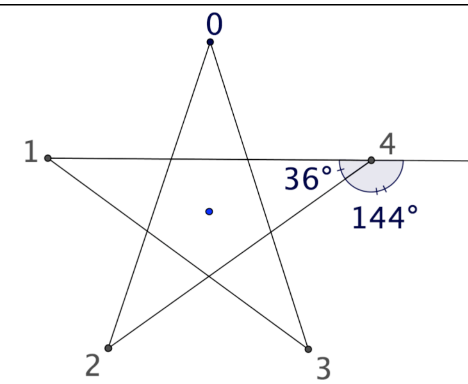


Figure 19. Finished star, showing an interior and exterior angle.

The reason for starting from 0, rather than 1, is that these numbers are remainders when you divide by 5. If you add 2 three times, you get 6. It takes only 5 to go around the circle.

$$6 \text{ steps} \div 5 \frac{\text{steps}}{\text{complete circuit}} = 1 \text{ complete circuit, remainder 1 step}$$

That is, 6 steps take you around the circle once, and a little more; you end at 1.

If you try a different pattern, such as adding 3 (0, 3, 1, 4, 2, 0), you might get a different shape. In this case, you get the same star, but traced in the opposite order. (Try it on a blank circle with only points marked.) If you add 4 (or subtract 1), you get a regular pentagon again. So the only symmetric star you can make for 5 equally spaced points is the usual 5-pointed one.

Find the interior and exterior angles for stars; try the methods above. However, think about how much of a total rotation you do if you follow the lines forming the star. One turn? more than one? It helps to draw a big star on the ground, and have someone carefully watch the total amount of turning as someone walks around the star.

Stars with other numbers of points

Use the circles attached at the end to try different "add n" patterns. Make copies of the basic circle to have a separate circle for each different star.

G. Exercises, problems, and projects

Exercises

1. Basic skills: Measure each of these angles in several ways:

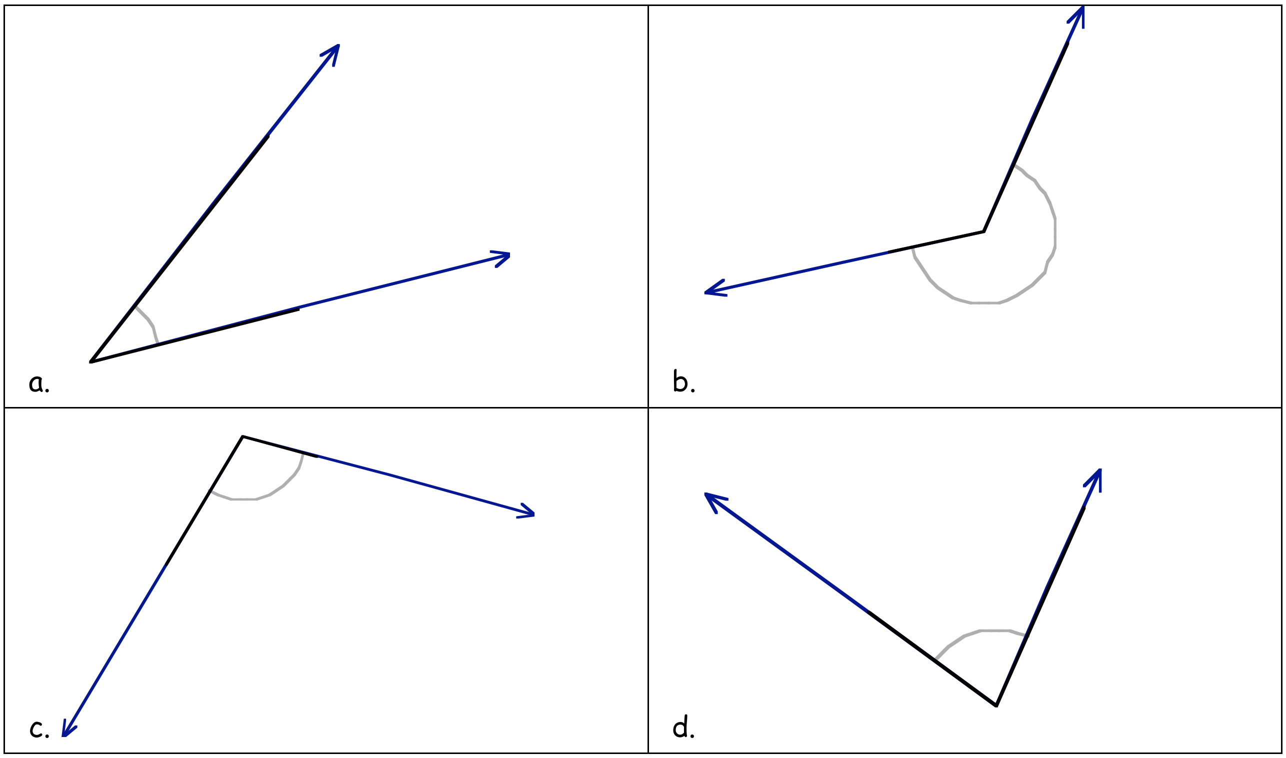
Visual estimation

With the square patty paper protractor

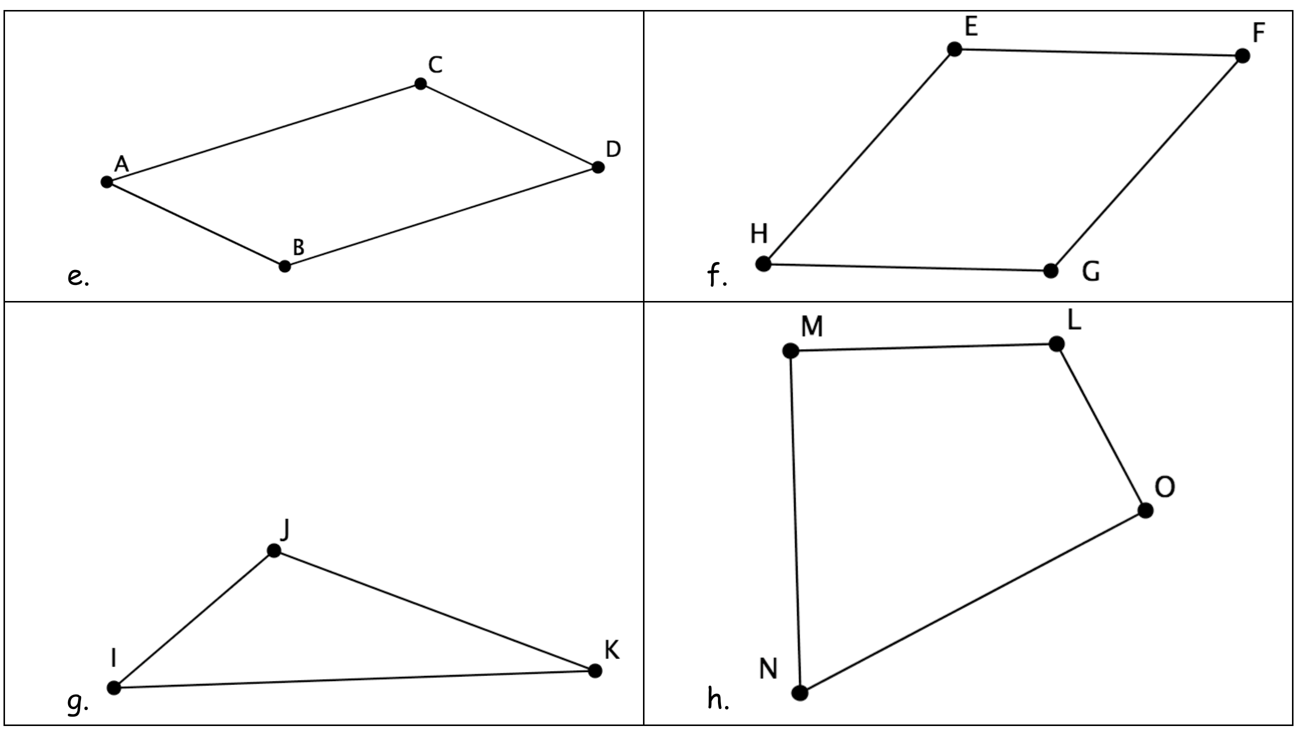
With the circular folded protractor

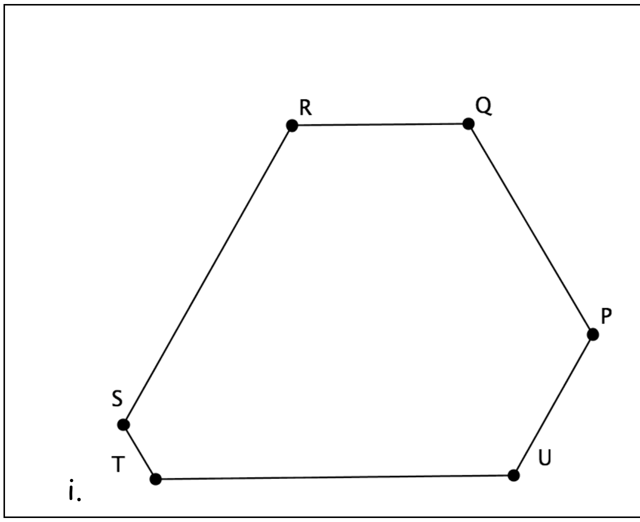
With a purchased protractor. It will help to use a ruler to extend the lines; these pictures are rather small.

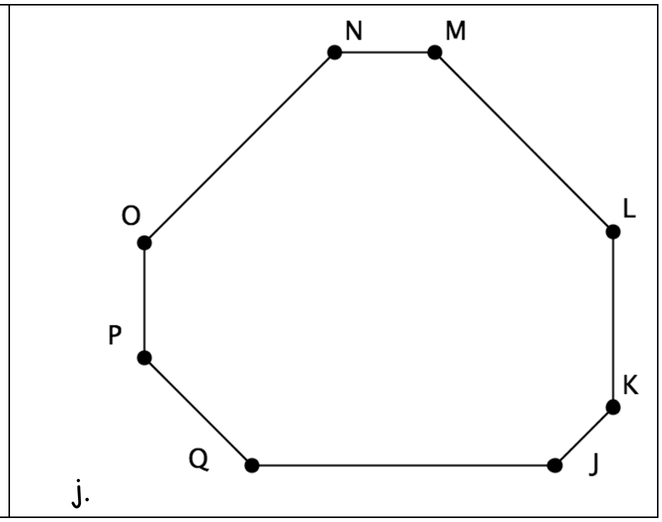
Use the results from b, c, and d to improve your visual estimating skills.



Estimate, then measure each of the interior angles in these polygons.







2. Basic skills/facts. What are the measures in degrees of the interior angles of each of these figures? You should be able to figure these out by reasoning about regular polygons without directly measuring.

   a. square

   b. rectangle

   c. equilateral triangle

   d. half an equilateral triangle (fold so that two sides match)

   e. half a square (along the diagonal)

   f. a regular hexagon

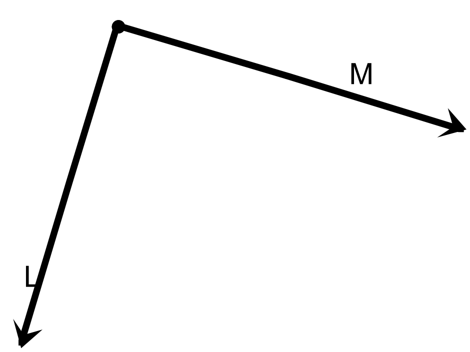
3. On a clock (not a digital clock!),

   a. How many degrees does the hour hand rotate in 1 hour? How many turns? (That is, what fraction of a turn?)

   b. How many degrees does the minute hand rotate in 1 hour? How many turns?

   c. How many degrees, and how many turns, does the hour hand rotate in a day?

4. What are the possible measures of this angle as a rotation? Detailed questions below.



a. Show all possible answers in two tables, as below. One table should show rotations that start in the direction of L, and turn towards M. Use positive numbers for counterclockwise rotations, and negative numbers for clockwise rotations.

b. Find the pattern of differences in each table.

Rotations from L to M

Turns	. . .								. . .
Degrees	. . .								. . .

Rotations from M to L

Turns	. . .								. . .
Degrees	. . .								. . .

Problems

Two other systems for measuring angles.

5. Make a second patty paper protractor like the first, except mark it with points of the compass:

N, S, E, W (that is, North, South, East, West)

NE, SE, NW, SW (that is, NorthEast, etc.)

NNE, ENE, ESE, SSE, SSW, WSW, WNW, NNW (North-NorthEast, etc.)

a. Use a compass and/or map to determine what direction north is where you are. Orient your patty paper protractor so that the directions are correct. Compass directions can be converted to angles from whatever reference direction you choose, such as north.

b. What number of degrees from north do each of the directions on the list above measure? Directions of rotation are given in this form: $45°$ east of north. This means to start by facing north, then turn towards the east with a rotation of $45°$.

6. Make a third folded circle protractor like the one in the class activity, except mark it like a clock. Now you can say things like "incoming at 2 o'clock" . In this system, you are giving an angle with respect to a reference direction, usually the direction that everyone in the conversation is facing.

7. What figures are traced by the following sequences of moves? Draw an accurate picture on graph paper. Use the side of one graph paper square as the unit of length.

a. repeat 4 [forward 3 right 90]

b. repeat 4 [forward 3 right 270]

c. forward 12 right 90
forward 4 right 90
forward 8 left 90
forward 4 right 90
forward 4 right 90
forward 8 right 90

8. What figures are traced by the following sequences of moves? Draw an accurate picture on isometric triangle or dot paper. Use the side of one graph paper triangle as the unit of length.

a. repeat 3 [forward 4 right 120]

b. repeat 10 [forward 4 right 60] (and do you really need to repeat 10 times?)

c. repeat 2 [forward 4 right 60 forward 4 right 120]

d. repeat 2 [forward 4 right 60]
forward 4 right 120
forward 8 right 120

e. repeat 2 [forward 4 right 30 forward 4 right 150]

9. Write Logo commands to trace each of the pattern block polygons. Decide on some length on the screen (such as 100) for the length of each of the short sides.

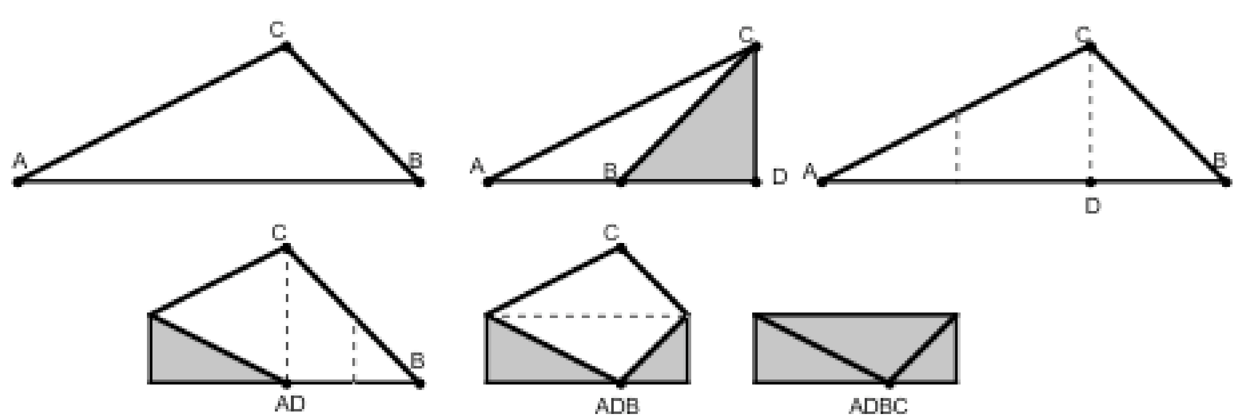
10. Interior and exterior angles of a triangle, again.

a. Using a straightedge, draw a triangle that takes up about half a piece of paper, and has three different angle measures. Cut it out and carefully trace a copy. Fold the cutout triangle as shown below. (The back of the paper is shaded.)

b. Position the triangle so the long edge is at the bottom. Mark letters A, B, C.

c. Make a fold through C so that the long edge is folded along itself. Unfold.

d. Fold A to D. Fold B to D. Fold C to D.



e. What does this say about the sum of the interior angles of your triangle?

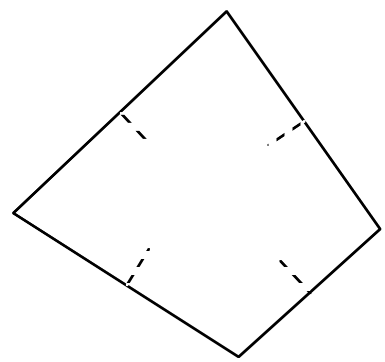
f. On your traced copy, label the vertices of your triangle with the same letters.

g. Measure the interior angles with a protractor, then add. How does this relate to part a, and the class activity on interior and exterior angles?

h. On the traced copy, measure the exterior angles with a protractor and add. How does this relate to part a, and the class activity on interior and exterior angles?

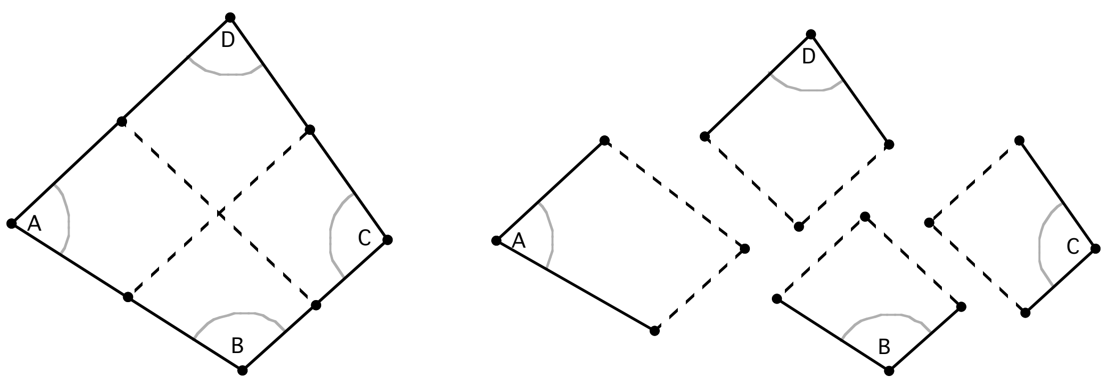
11. Interior and exterior angles of a quadrilateral, again.

a. Start with a quadrilateral with unequal sides and angles (though this works with *any* quadrilateral.) Cut it out.



b. Fold each side in half and make a pinch to mark the midpoint. Don't fold all the way across, just the side.

c. Mark the four interior angles A, B, C, D so that you can tell both which angle was the corner, and which side was the front. Draw lines connecting opposite midpoints, then cut along the lines to make four pieces, each with one of the angles A, B, C, D.



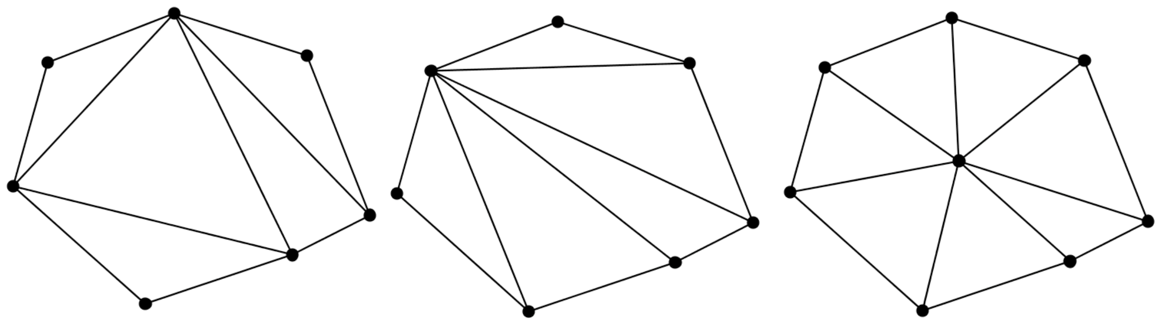
d. Physically add the angles A, B, C, D by putting the pieces together. Hint: don't flip any pieces over! Match equal side lengths. What kind of a shape do you get?

e. If possible, prove that you will always get that kind of shape, no matter what kind of quadrilateral you start with, by thinking about the various angles in this construction.

12. Sum of interior angles of a polygon, again. Make several examples of irregular polygons with varying numbers of sides.

    a. Divide the interior of each polygon into triangles. The sum of the interior angles of each triangle is 180° . Add all the angles of all the triangles in the polygon. How is this related to the sum of interior angles in the polygon? Explain.

    b. Here are three ways to divide a polygon into triangles. Do they give the same results for the sum of interior angles in the polygon? Why?



    c. After doing a number of examples, give a formula or method for finding the sum of the interior angles in a polygon with N sides, without knowing anything else about the polygon. Explain your reasoning.

13. Angles formed by intersecting lines. This is a topic from high school geometry, burdened with lots of terminology. In this problem, you will use transformations to illustrate why various angles are equal. Two copies of the figure, in dark marker, on thin paper, will make this easier to do physically. Or draw the top layer on a transparency.

More terminology from high school geometry:

Vertical angles are two non-adjacent angles formed by two intersecting lines. (*Adjacent* angles share a side.)

    a. What transformation will place angle 1 on top of angle 3, and vice versa? What transformation will put angle 2 on angle 4? Be specific: give a center and an angle for a rotation, draw a vector for a translation, draw a line for a reflection. If you can superimpose the angles, this shows that vertical angles are equal.

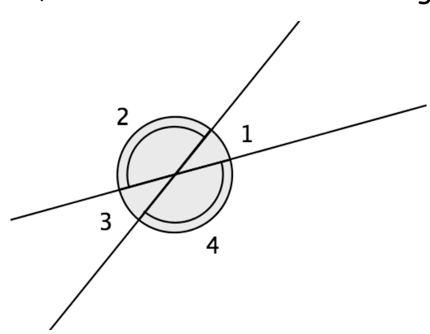


Figure 20. Two intersecting lines form two pairs of vertical angles.

If two parallel lines are crossed by another line, that line is called a *transversal.* Many angles are formed, and many of these pairs are equal, and the others are closely related.

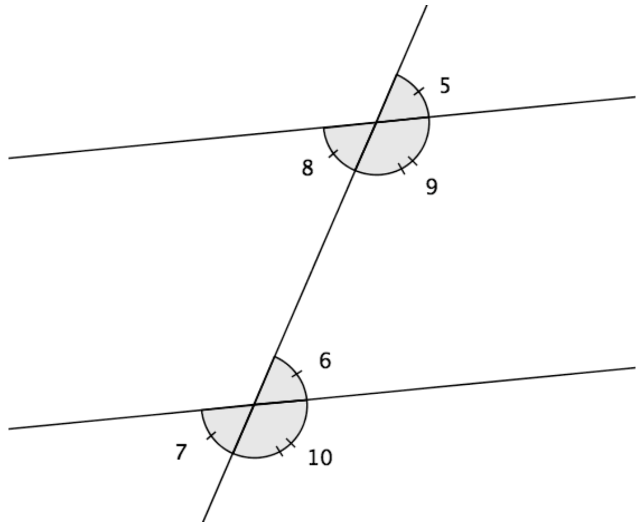


Figure 21. A transversal cutting two parallel lines forms many pairs of equal angles.

b. *Corresponding angles* are two angles that are both above, or both below, the parallel lines, and on the same side of the transversal. Find a specific transformation that will put angle 6 on angle 5, showing that these corresponding angles are equal. Find the other three pairs of corresponding angles, and find the transformations that shows that members of each pair are equal.

Interior angles are between the parallel lines, and on the same side of the transversal.

c. *Alternate interior angles* are between the parallel lines, and on opposite sides of the transversal. Find a specific transformation, or sequence of transformations, that will put angle 6 on angle 8, showing that these alternate interior angles are equal. Find the other pair of alternate interior angles, and find the transformation that shows they are equal.

Exterior angles are on the outside of the parallel angles, and on the same side of the transversal.

d. *Alternate exterior angles* are on the outside of the parallel angles, and on the opposite sides of the transversal. Find a specific transformation, or sequence of transformations, that will put angle 5 on angle 7, showing that these alternate exterior angles are equal. Find the other pair of alternate exterior angles, and find the transformation that shows they are equal.

e. Use the previous parts of this problem, and adjacent angles, to reason that interior angles are supplementary.

f. Use the previous parts of this problem, and adjacent angles, to reason that exterior angles are supplementary.

14. Remember that a *regular polygon* is a polygon has all side lengths equal and all angles equal.

Use the class activity to figure out the measure of *one* exterior, interior, and central angle in a *regular* polygon. Fill in the table below and find a pattern. The last line should give a formula or method that makes it easy to figure out any angle in a regular polygon with any number of sides.

Use the pattern or formula to make a regular polygon with more sides (such as 9, 10, or 12.) Check that all the angles are correct, and that the polygon is regular.

Number of angles	Sum of exterior angles	One exterior angle	One interior angle	One central angle
3				
4				
5				
6				
7				
8				
N (any whole number)				

15. Regular polyhedra and regular tessellations. Uses the results of the previous problem.

A *tessellation* of the plane is covering the plane with polygons with no gaps or overlaps. Recall that in the regular polyhedron activity, when you use congruent shapes, there are three choices for what happens at a vertex: the polygons pop up to form a 3-dimensional corner, the polygons lie flat with no gaps, or there are too many to lie flat, or pop up. In the first case, you have a chance of continuing to make a polyhedron. In the second case, you have a chance of continuing to make an infinite tessellation of the plane.

Use the results of the previous problem to figure out all possibilities for regular polyhedra and tessellations made with one kind of regular polygon. Describe the polyhedra, and draw an accurate picture of each tessellation. Give calculations and explanations supporting your conclusions. Include the different possible numbers of polygons at each vertex.

Example: N=4 (squares), interior angle at each vertex is 90°

1 or 2 at a vertex: won't make anything

3 at a vertex: polyhedron (cube) $90^\circ + 90^\circ + 90^\circ = 270^\circ$, which is less than 360°

4 at a vertex: tessellation (checkerboard): $90^\circ + 90^\circ + 90^\circ + 90^\circ = 360^\circ$

5 or more: neither polyhedron nor tessellation: sum of angles is more than 360°

Note: Only consider those that are *edge to edge:* no vertex of a face can connect to the middle of an edge of another face. The brick pattern shown is not an edge-to-edge tessellation.

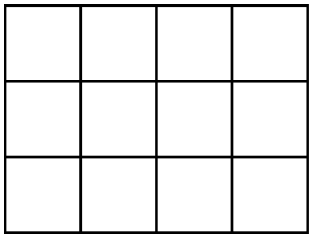


Figure 22. An edge-to-edge tessellation with squares

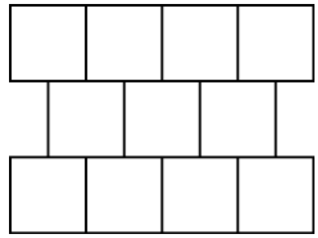


Figure 23. A tessellation with squares that is not edge-to-edge

16. Semi-regular tessellations. Uses the results of the previous two problems.

A *semi-regular* tessellation uses regular polygons, is edge-to-edge, and has the same combination of polygons at each vertex. However, you may use more than one kind of regular polygon (such as squares and triangles.)

a. Use results of previous problems to find what combinations of regular polygons could meet at a vertex and lie flat.

b. Then try to make each tessellation (most of them do work, but it's possible that some won't continue past a few pieces.) Obtain or make templates for each of the possible regular polygons, making sure the edge length is the same is all of them. Then trace them to make the tessellations.

c. Also give the vertex symbol for each successful tessellation. (For example, the checkerboard above is 4.4.4.4, since four 4-sided polygons meet at each vertex.)

17. Semi-regular polyhedra. Uses the results of the previous three problems. (This problem is quite challenging.)

A *semi-regular* polyhedron uses regular polygons, is edge-to-edge, and has the same combination of polygons at each vertex. However, you may use more than one kind of regular polygon (such as squares and triangles.)

a. Use results of previous problems to find what combinations of regular polygons could meet at a vertex and pop up.

b. Then try to make each tessellation (most of them do work, but it's possible that some won't continue past a few pieces.) Use snap-together plastic pieces if at all possible. Look up the name of each successful polyhedron on the Internet or in a book.

c. Also give the vertex symbol for each successful polyhedron. (For example, the cube is 4.4.4, since three 4-sided polygons meet at each vertex.)

18. Concave polygons. Describe what happens if you make a non-convex figure with rope and walk around it. What can you say about the total rotation if the figure doesn't cross itself?

19. Regular and star polygons by hand, following Example 2, using compass, protractor, and straightedge. Make a number of copies of the marked circle so that each different shape is on a separate sheet.

a. Example 2 made a regular pentagon and a regular 5-pointed star. Use the same method to make a regular 7-gon and all possible regular 7-pointed stars.

b. Figure out the exact measure of the following angles, and explain your reasoning:

any interior angle at a vertex of the regular 7-gon

the sum of the interior angles of the regular 7-gon

any exterior angle at a vertex of the regular 7-gon

the sum of the exterior angles of the regular 7-gon

Also measure with a protractor to make sure your predictions match what you made.

c. For **each** of the different star polygons you made, figure out the exact measure of the following angles, and explain your reasoning:

any interior angle at a vertex of the star polygon

the sum of the interior angles of the star polygon

any exterior angle at a vertex of the star polygon

the sum of the exterior angles of the regular 7-gon

Also measure with a protractor to make sure your predictions match what you made.

20. Make the same figures as in the previous problem with a Logo program, using what you found about the angles. See the Logo section, or the class Moodle page, for the URLs of some online and downloadable Logo programs. To show your work:

Windows: Alt-Print screen, then paste into a word processing document, or just print it on paper

Macintosh: Apple key-Shift-4, then outline the part of the screen you want. It will be saved as a file called Picture 1, etc. Paste or print this picture.

If it didn't work, or your picture is all black, you can just list the Logo commands you used to make the picture, and describe the picture.

21. General results about regular polygons and/or star polygons. Repeat the parts of the previous two problems with some number other than 7. Preferably, do several examples and find a pattern that will help answer the general questions below.

a. What is the measure of one interior angle in a regular K-gon (polygon with K equal sides)? One exterior angle?

b. What are Logo commands that will draw a regular K-gon with a specified side length?

c. What is the measure of one interior angle in a regular K-pointed star? One exterior angle? (In general, there are several different stars for each K.)

d. What are Logo commands that will draw a regular K-pointed star with a specified side length? (All line segments drawn should have the same length.)

e. How can you choose numbers N and A to ensure that the Logo program

```
repeat N [right A forward F]
```

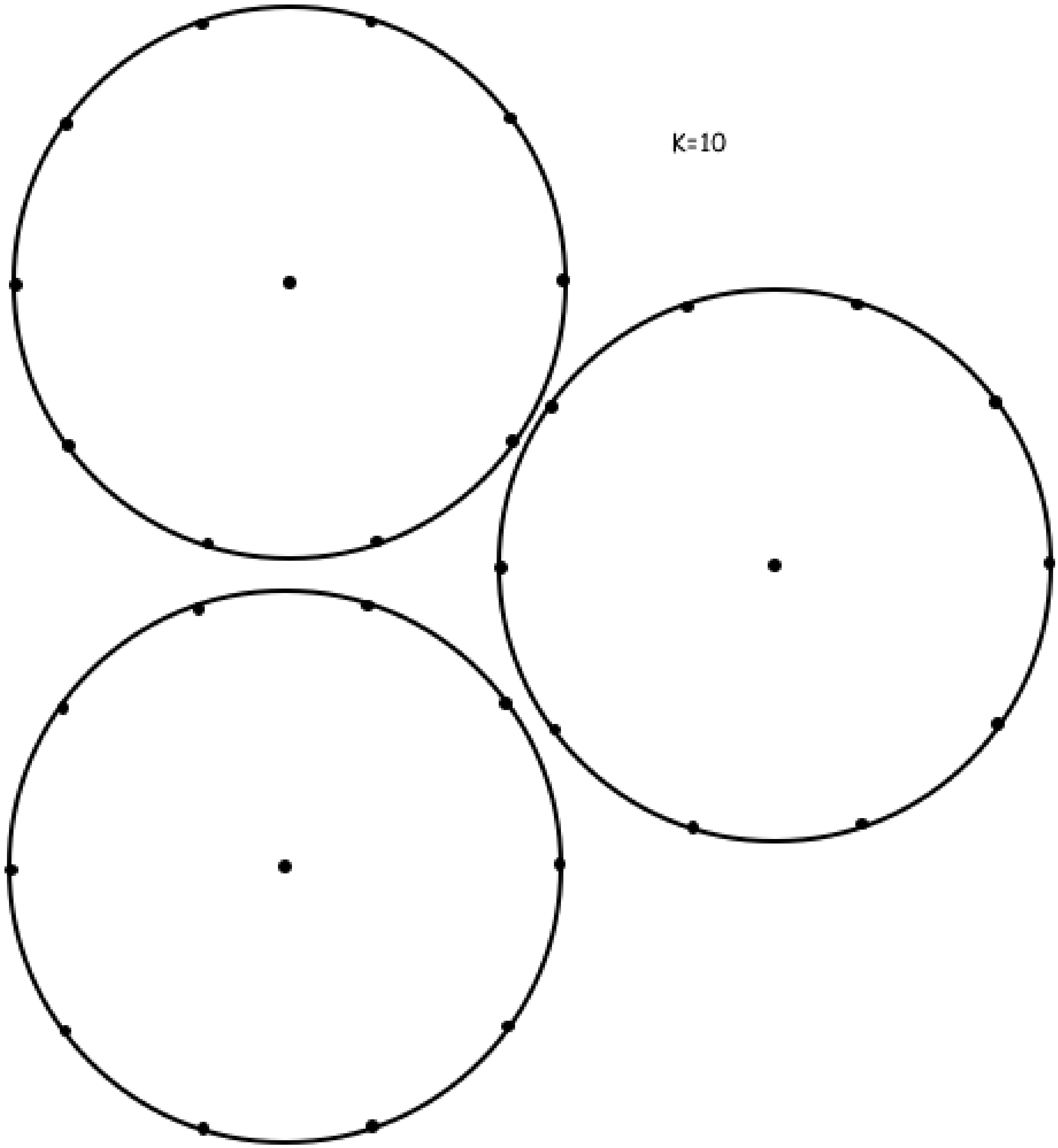
will draw a polygon or star polygon that ends at its starting point? How many vertices will it have?

Hint: think of angles as measured in fractions of a turn, such as 1/3, 4/7, 3/16, etc. You will have to convert these to degrees to use them in Logo.

22. Clock angles.

a. Develop a method (or formula?) for finding the angle between the hour and the minute hand of a clock at any specified time. Example: the angle between the hands at 3:00 is 90 degrees. The angle at 3:15 is *not* 0 degrees, because the hour hand has moved to a quarter of the way between 3 and 4, while the minute hand is at 3.

b. At what times is the angle between the hands equal to 0 degrees? To 180 degrees? Can you find a method or formula to get the times when the hands have some specified angle?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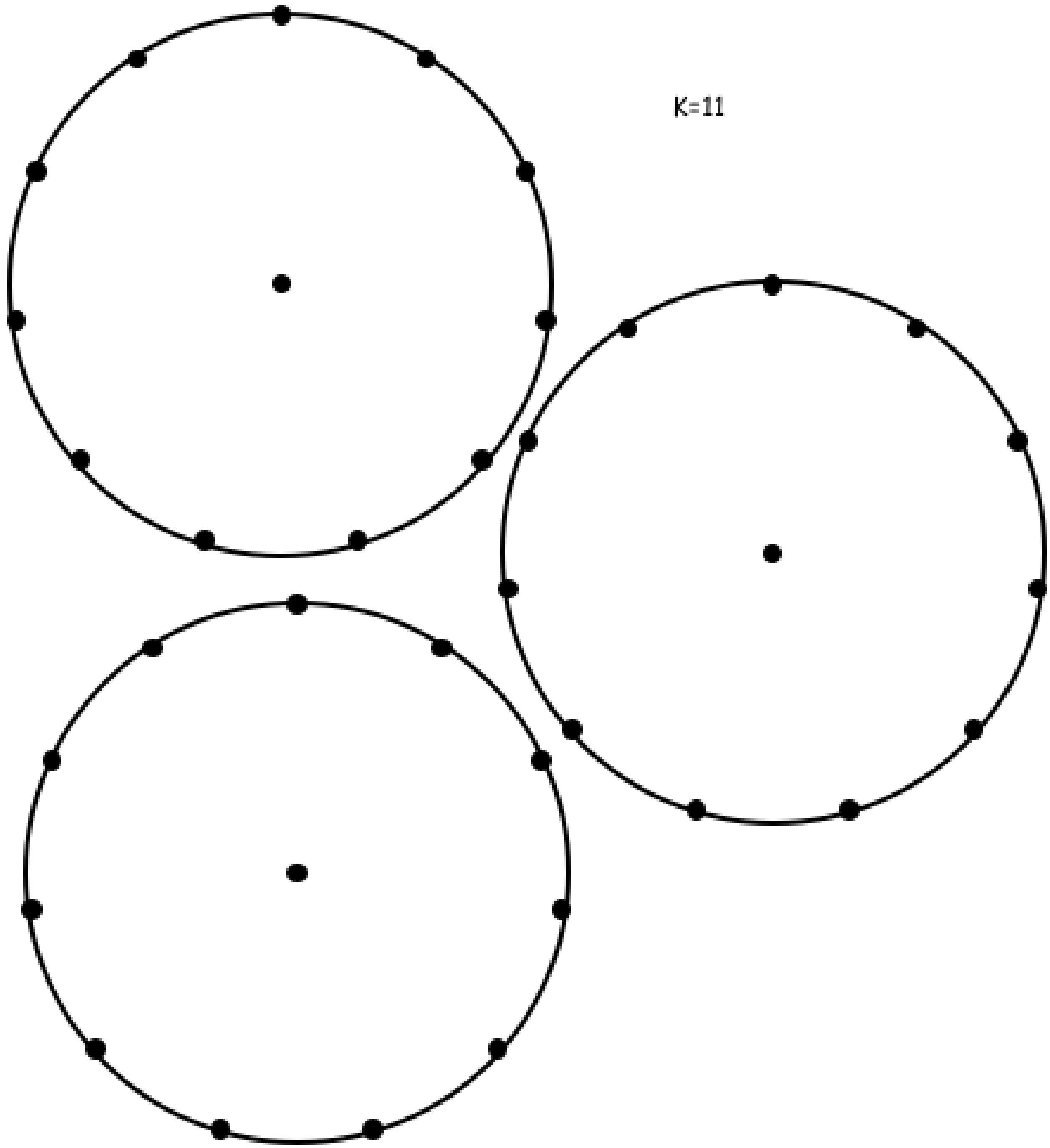
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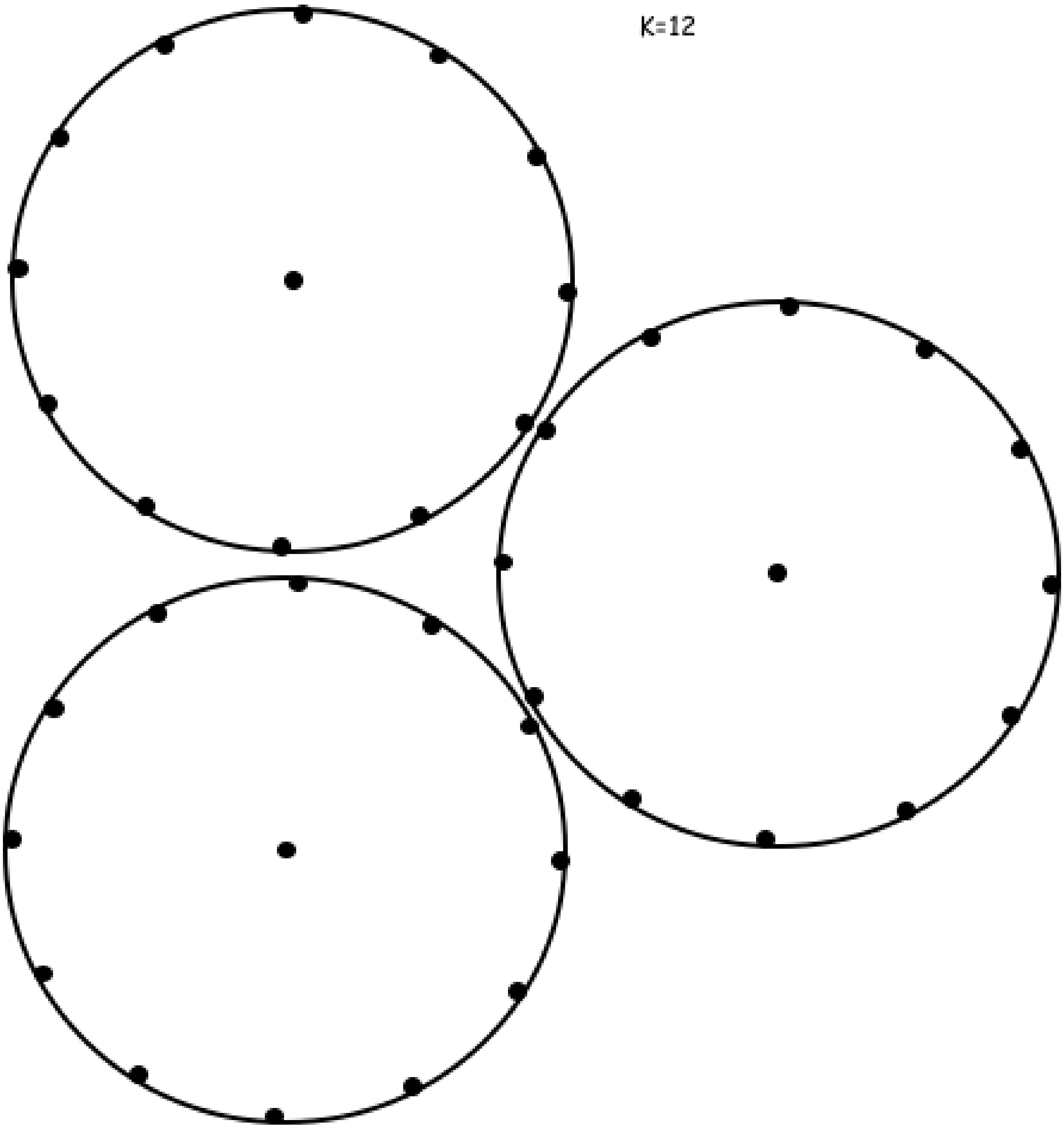
K=12