Descartes's Logarithm Machine

In the *Geometry* (1952), Descartes considered the problem of finding *n* mean proportionals (i.e. geometric means) between any two lengths a and b (with $a < b$). That is, the problem is to find a sequence of lengths, beginning with a and ending with b, such that the ratio of any two consecutive lengths is constant. Hence the terms of the sequence have a constant ratio of r, and form a geometric sequence beginning with a and ending with b.

In modern algebraic language, the problem is to find a sequence $x_0, x_1, \ldots, x_{n+1}$ such that for some fixed ratio r, $x_k = a \cdot r^k$, and $x_{n+1} = b$. That is, *n* mean proportionals between *a* and *b* are *n* numbers between *a* and *b*, not counting *a* and *b* themselves. For a modern algebraist, solving this problem involves finding a (n+1)st root:

$$b = a \cdot r^{n+1}$$

$$r = \sqrt[n+1]{\frac{b}{a}}$$

A. Descartes' linkage machine from the *Geometry*

Book II of the *Geometry* shows the following illustration.

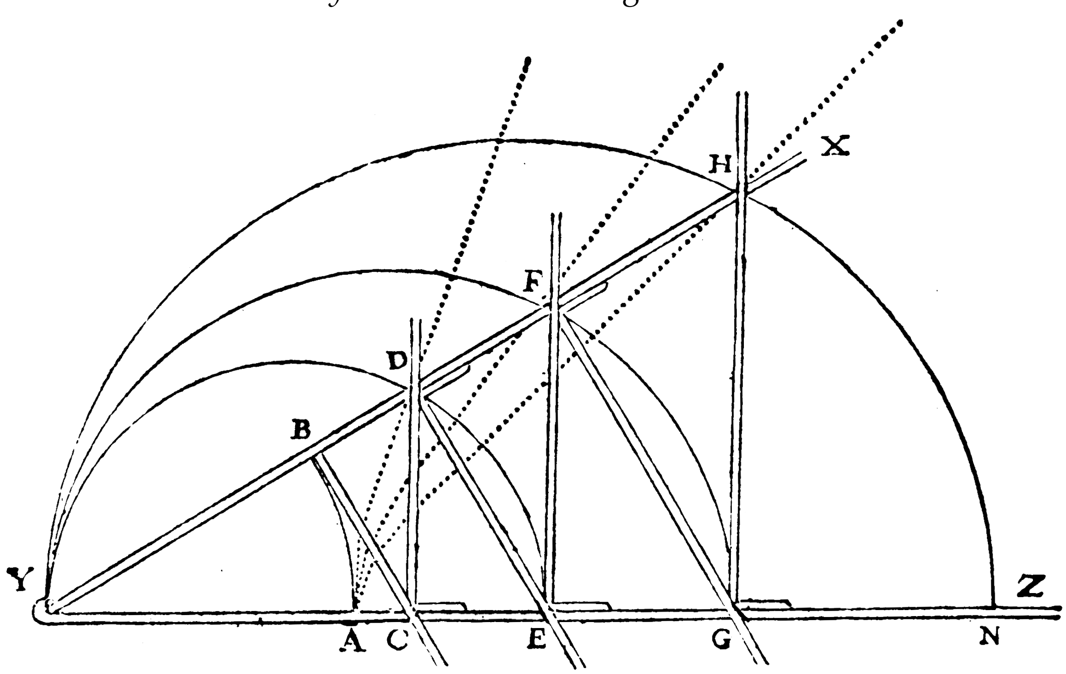


Figure 1. Descartes's linkage machine.
[Applet: DescartesLogPic.html]

Descartes describes it as a linkage of (an infinity of) hinged rulers. The angle between rulers YZ and YZ can be adjusted, as can the position of point B on ruler YX. As the angle XYZ changes, rulers BC, DE, FG, etc., remain perpendicular to YX, and rulers CD, EF, GH, etc. remain perpendicular to YZ, and the positions of points C, D, E, etc., move to keep these right angles. As the angle XYZ changes, point B traces a circle. Point D traces a more complicated curve AD; similarly, points F, H, etc. trace other, still more complicated, curves.

For the rest of Book II, Descartes argues that processes like these should be considered as legitimate mathematical ways to define curves, rather than being restricted to the lines and circles (straightedge and compass constructions) of ancient Greek mathematics. Then he describes equations for the curves, and classifies the complexity of the curve by the algebraic degree of the equation.

Example 1. Finding an equation for curve AD. (Following the work of translators Smith and Latham. Descartes left the work to the reader.)

Call YA=YB=a, YC=x, and CD=y.

Using similar triangles ($\Delta YCD \sim \Delta YBC$)

$$\frac{CD}{CY} = \frac{BC}{BY} \text{; that is, } \frac{y}{x} = \frac{BC}{a}$$

Since the triangles are right triangles,

$$BC^2 + YB^2 = YC^2 \text{, so } BC^2 = x^2 - a^2 .$$

Squaring the proportion and substituting gives

$$\frac{y^2}{x^2} = \frac{x^2 - a^2}{a^2}$$

Cross multiplying gives a polynomial equation of degree 4 that relates x and y.

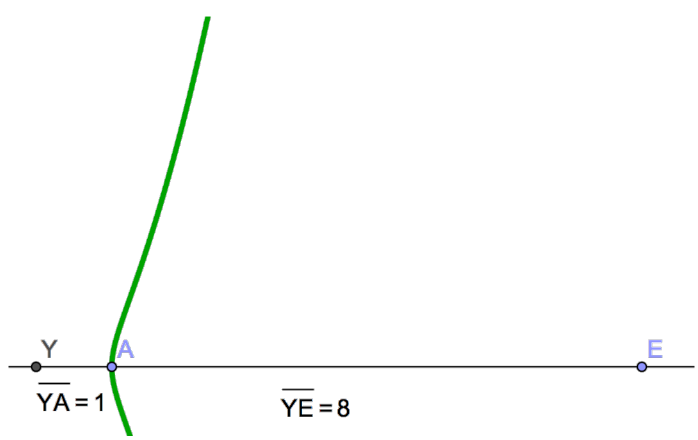
$$a^2 y^2 = x^2 (x^2 - a^2)$$

Try this (1). Find equations for curves AF and AH.

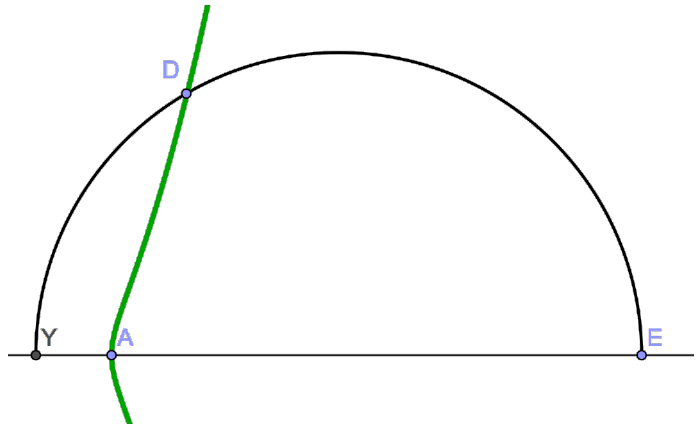
In Book III Descartes returns to this mechanism, and uses it to solve the mean proportionals problem geometrically. See the applet Descartes2MeanProp.html. This is what he says:

To find two mean proportionals between YA and YE:

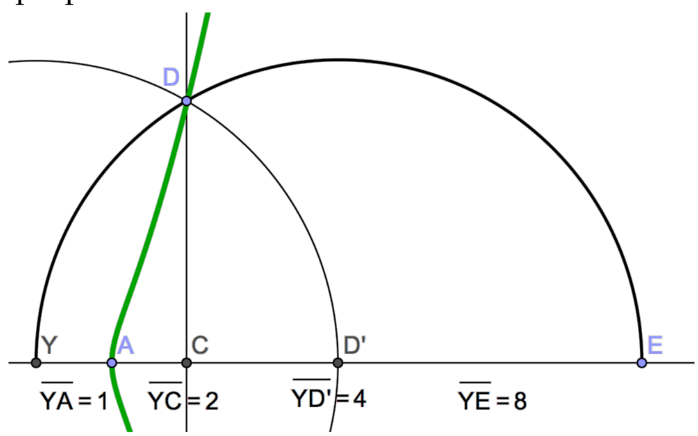
Set the machine to start at A and draw the curve AD in Figure 1. Mark the point E.



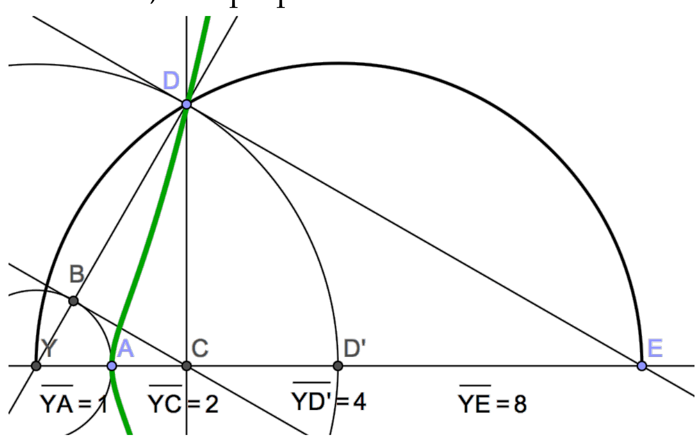
Construct a circle with diameter YE. Mark the intersection point, D, of the curve and the circle.



Use a circle with center Y to mark distance YD on line YE. Distance YD is one of the mean proportionals. Drop a perpendicular from D to line AE. The intersection point, C, is the other mean proportional.



To see the rest of the relevant lines (and the right triangle for a geometric mean) from Figure 1, draw in line YD, and perpendiculars to YD and YE.



Why are these the required mean proportionals?
Here is a version of Figure 1 with only the lines.

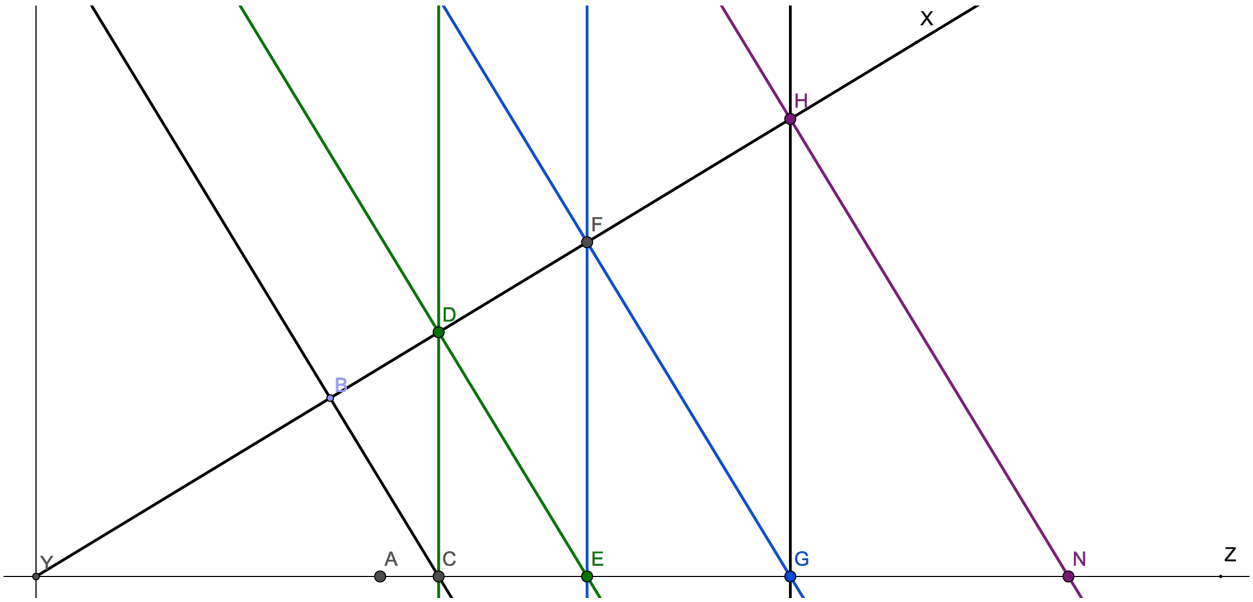


Figure 2. Same construction as Figure 1, but lines only.

There is a sequence of similar right triangles (see the geometric mean construction in the Lecture Notes [Sim&GM.pdf](Sim&GM.pdf))

$$\Delta YCB \sim \Delta YDC \sim \Delta YED \sim \Delta YFE \sim \ldots,$$

thus the proportions

$$\frac{YC}{YB} = \frac{YD}{YC} = \frac{YE}{YD} = \frac{YF}{YE} = \cdots$$

If we set $r = \frac{YC}{YB}$, then

$$YC = rYB, \quad YD = r^2 YB, \quad YE = r^3 YB, \text{ etc.}$$

That is, the lengths form a geometric sequence.

Descartes, after stating that, "there is, I believe, no easier method of finding any number of mean proportionals, nor one whose demonstration is clearer," (1952, p. 155) goes on to criticize his own construction for using curves of a higher class than is necessary. Finding two mean proportionals, for example, is equivalent to solving a cubic equation, and can be accomplished by using only conic sections (first class), while the curve traced by D is of the second class. The solution of cubics by intersecting conics had been achieved by Omar Khayyam, and was well known in seventeenth century Europe (Joseph, 1991, Berggren 1986). Descartes spent much of the latter part of the *Geometry* discussing the issue of finding curves of minimal class (that is, degree) which will solve various geometry problems (1952).

Descartes expounded an epistemological theory which sought a universal structural science of measure which he called "mathesis universalis". Fundamental to his program was his classification of curves in geometry. He wanted to expand the repertoire of curves that were allowed in geometry beyond the line and circle, but he only wanted to include curves whose construction he considered to be "clear and distinct". For Descartes this meant curves which could be drawn with linkages and classified by his system according to pairs of algebraic degrees. These curves he called "geometrical" and all others he called "mechanical."

This distinction is equivalent to what Leibniz would latter call "algebraic" and "transcendental" curves. Descartes viewed "mechanical" (i.e. transcendental) curves as involving some combination of incommensurable actions. Examples that he specifically mentioned are the spiral, quadratrix, and cycloid. These curves all involve a combination of rotation and linear motion that cannot be connected and regulated by some linkage. The drawing of such curves involves rolling a wheel or the unwinding of string from a circle.

B. The graph of the logarithm function

Next we'll use Descartes's machine to construct points on the graph of a logarithm function: $y = \log_b x$. See the applet DescartesGraph.html.

Descartes did not think of curves in this way. He used the mathematics of his machine to solve a related problem posed by a correspondent. The construction that follows is not in Descartes; it is a repurposing of the machine, combined with the type of thinking that was historically used to construct logarithm tables. The result is a geometric version of a logarithm table, constructed without computations.

A logarithm table is constructed by pairing an arithmetic sequence (generated by repeatedly adding a fixed number) with a geometric sequence (generated by repeatedly multiplying by a fixed number), then interpolating to fill in the gaps. (See the SlideRules.pdf Lecture Notes, if you haven't already done it.) Since Descartes's machine constructs a geometric sequence between two values, it can interpolate any finite number N of subdivisions between two values in the geometric sequence column. The arithmetic column can be easily subdivided geometrically in the construction.

Here's how to build a simulation of Descartes' device for the construction of geometric sequences with $a = 1$ and ratio *r.* A dynamic geometry program is recommended, though you can use straightedge, compass, and squares.

Let O be the origin of a rectangular coordinate system, and let H be any point on the unit circle. Construct ray $\overrightarrow{OH}$. Construct perpendiculars alternately to $\overrightarrow{OH}$ and the *x* axis, giving points H, H_2, H_4, H_6, etc., on $\overrightarrow{OH}$, and G_1, G_3, G_5, G_7, etc., on the *x* axis. By the previous reasoning, the distances G_1, H_2, G_3, H_4, … form a geometric sequence.

By moving H around the circle, the distances of the labeled points from the origin will form geometric sequences with any common ratio. That is, if $r = OG_1$, then $r^2 = OH_2$, $r^3 = OG_3$, $r^4 = OH_4$, etc.

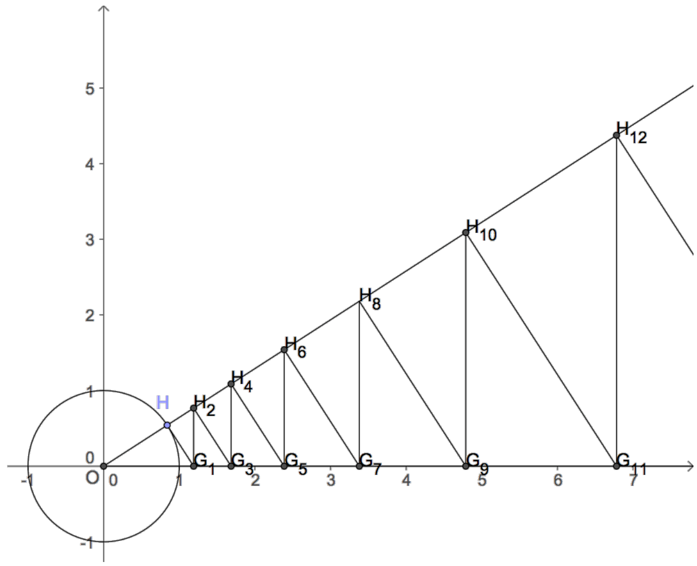


Figure 3. Construction of positive powers of r.

This construction can also be extended to the interior of the unit circle to obtain segments whose lengths are the negative powers of r. Once again, as with the preceding construction, the odd powers of r are on the horizontal while the even powers of r are on ray $\overrightarrow{OH}$.

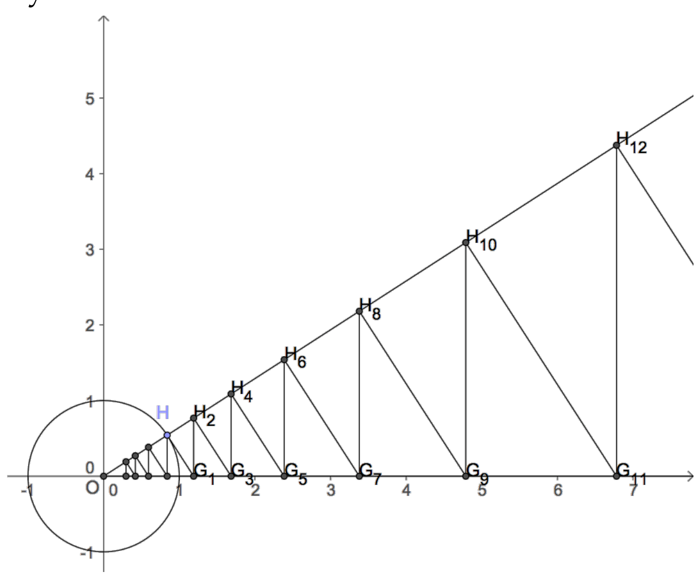


Figure 4. Construction of negative powers of r.

Use circles centered at O to mark the H distances on the x axis.

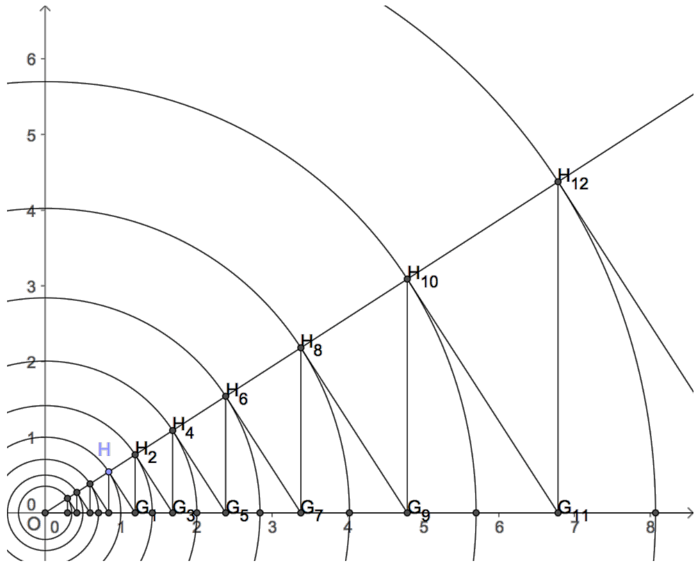


Figure 5. All powers of r marked on the x axis.

Name the intersection points on the *x* axis

$$\ldots, G_{-6}, \quad G_{-4}, \quad G_{-2}, \quad G_0 = 1, \quad G_2, \quad G_4, \quad G_6, \ldots,$$

with the indices of the corresponding H's. Thus we now have a geometric sequence, $G_i = r^i$, laid out on the x-axis, whose common ratio or density can be varied as the point H is rotated.

In order to construct logarithmic curves, we must now construct an arithmetic sequence $\{A_i\}$ on the y-axis with a variable common difference. In the diagram, these points are determined by the point A_1 on the *y* axis. For each pair of points, G_i, A_i, with G_i on the *x* axis and A_i on the y axis, intersect a horizontal line at A_i with the vertical line at G_i to obtain a point on the logarithm curve. The resulting points have been connected with line segments, forming a piecewise linear approximation to a graph.

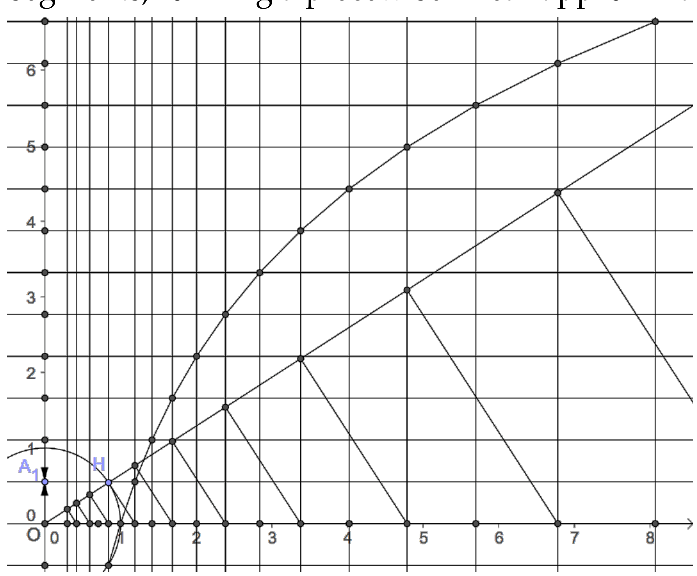


Figure 6. An approximation to a logarithm curve.

Note that the horizontal and vertical lines form semilog graph paper: the scale in one direction is equally spaced, and in the other the spacing is logarithmic. Before the advent of computer graphing programs, this type of paper was used by scientists and others to try to find exponential and logarithmic relationships in data.

This construction yields an adjustable curve. By moving H around the unit circle, or A_1 along the y axis, one can map any geometric sequence against any arithmetic sequence to obtain points on the graph of any logarithm function. In Figure 7, the point H is adjusted so that G_1 is at 2 on the x axis, and the point A_1 is at 1 on the y axis. Hence the points lie on the graph of the log base 2.

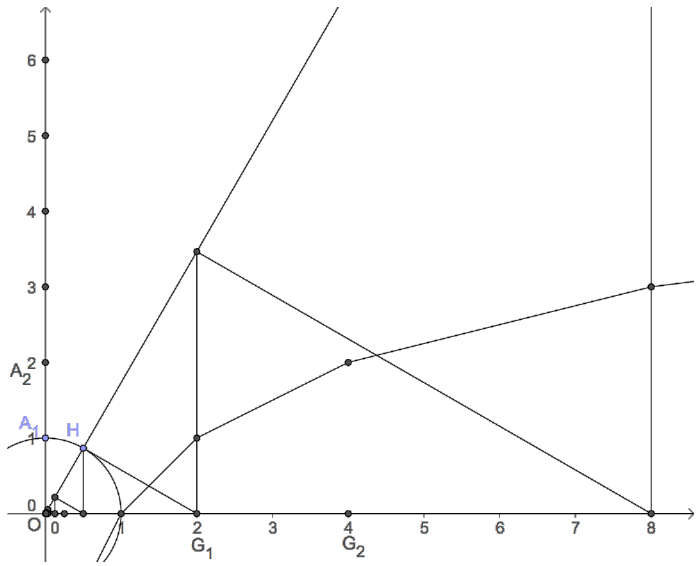


Figure 7. Widely spaced points on the graph of $y = \log_2 x$.

Readjusting A_1 and H so that A_2=1 and $G_2 = 2$, the points are still on the graph of $y = \log_2 x$, but are more densely spaced.

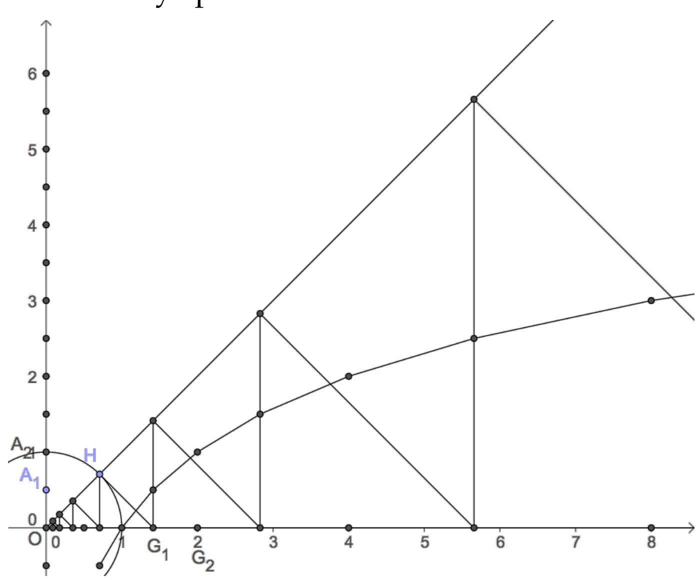


Figure 8. More densely spaced points on the graph of $y = \log_2 x$.

Try this (2). Explore this logarithm curve in more detail:

a) Figure out how to adjust H and A_1 for specific logarithm curves.
b) How can you get points on the graph of $y = \log_3 x$?
c) How can you get points on the graph of $y = \log_{10} x$?
d) How can you get points on the graph of $y = \log_b x$ for any positive b?
e) If you have points on $y = \log_b x$, how can you double the density of the points?
f) Find the angle that OA makes with the x axis when H is at 2 and A_1 is at 1.
g) Find the base of the logarithm function when OA makes an angle of $45°$ with the x axis.
h) Find relationships between the angle that OA makes with the x axis, the coordinates of H and A_1, and the base of the resulting logarithm function.

C. The derivative of the logarithm function

See the applet LogSlope.html.

After watching these log curves shift and bend dynamically, you can begin to look carefully at the slopes between points on the curves. Several interesting patterns come to light. Suppose you want to use the slopes between constructed points to approximate the tangent slope at a point, say, for example, at (1,0). It is visually apparent that using the point (1,0) in the calculation is not the best thing to do. The slope between the nearest points to the right and left gives a better approximation of the tangent slope. This is true for most curves, not just the logarithm. Here, at (1,0), we want to calculate the secant slope between G_{-1} and G_1. Letting r equal the common ratio of the geometric sequence, and d equal the common difference of the arithmetic sequence, the slope k at (1,0) is

$$k = \text{slope at } (1,0) = \frac{2d}{r - \frac{1}{r}} = \frac{2rd}{r^2 - 1}.$$

Now approximate the slope at any other point on the constructed curve. This approximate slope at $(G_n, A_n) = (r^n, nd)$, is found by computing the secant slope between G_{n-1} and G_{n+1}. The calculation yields:

$$\text{slope at } (G_n, A_n) = \frac{2d}{r^{n+1} - r^{n-1}} = \frac{1}{r^n} \cdot \frac{2rd}{r^2 - 1} = \frac{k}{r^n}.$$

Here we have the approximate tangent slope at a point on a logarithm written as 1 over the x value times a constant. The constant k is the slope of the curve at (1,0). Of course these slopes are all approximations, but once the slope at (1,0) is approximated it can be divided by the x-coordinate at any other point to get the corresponding slope approximation at that point. By making the constructed points on the curve denser the approximations all improve together at the same rate. Thus the essential derivative property of logarithms is revealed without recourse to the usual formalisms of calculus. In fact, even more is being displayed here than the usual derivative of a logarithm. One sees that the all the slope approximations converge uniformly as the density of the constructed points is increased.

It is strange that when the derivative is developed in calculus classes, it is defined using secant slopes from the point in question, rather than around the point. It would seem that nobody is directly interested in secant slope approximations, except as an algebraic device from which to define a limit. The practical geometry of secant slopes is ignored.

D. More exercises

The idea of interest in finance goes back to the ancient Babylonians.

Try this (3). Suppose you will need $5,000 a year from now, and have $4,000 to save now.

a) If interest is compounded monthly, what rate would you need to have $5000 by the end of the year?

b) Explain what compound interest has to do with logarithms and mean proportionals.

E. References

These lecture notes are adapted from

Dennis, D. & Confrey, J. (1997). Drawing Logarithmic and Exponential Curves with the Computer Software Geometer's Sketchpad: A Method Inspired by Historical Sources. In J. King & D. Schattschneider (Eds.), Geometry Turned On: Dynamic Software in Learning, Teaching and Research. pp. 147-156. Washington D.C.: Mathematical Association of America. http://www.quadrivium.info/mathhistory/Log.pdf

Descartes, R. (1925). *The geometry of Rene Descartes*. (D. E. Smith & M. L. Latham, Trans.). Chicago: Open Court Publishing Company. From http://www.archive.org/details/geometryofrene00desc.