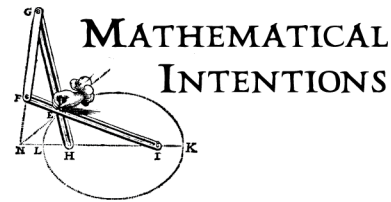


Apollonius and Conic Sections



A. Some history

Apollonius of Perga (approx. 262 BC–190 BC) was a Greek geometer who studied with Euclid. He is best known for his work on cross sections of a cone.

The mathematicians of the 17th century all read Apollonius. Often original works of mathematics from this period were written as commentaries on the work of Apollonius or as reconstructions of supposedly lost works of Apollonius. When it came to the study of curves, they often saw themselves as attempting to extend and generalize the work of Apollonius, especially *The Conics*. Important Arabic additions to *The Conics* from the tenth and eleventh century were also crucial to the mathematics of the scientific revolution. In fact, even to this day, large parts *The Conics* of Apollonius do not exist in their original Greek form, and are known to us only through Arabic translations.

The Conics is written in Euclidean mathematical style: postulates, theorems, and proofs, and verbal descriptions of the objects. The geometry is of static, not moving objects. There was no algebraic notation, no coordinates, and, in fact, no figures.

The Conics is a systematic approach to conic sections which takes the view of a conic curve in the plane as a set of parallel, ordered line segments from an axis. This approach is very close to the idea of coordinate geometry. For this reason certain concepts developed in *The Conics* were fundamental in the development of the function concept. *The Conics* contains a variety of beautiful theorems that are very little known today. In particular, the theorems on symmetry, axes and conjugate diameters could be studied at the high school level; the main mathematical results used are similarity and geometric means. These theorems were used extensively throughout the seventeenth century, for example in Newton's *Principia* (1687) for the discussion of planetary orbits.

B. Basic facts about cones

The rationale for studying cones was that Euclid's geometry dealt with lines and circles in a plane.

Definition of a cone

A cone, for Apollonius, is the shape generated by considering all lines in space that pass through a fixed circle and a fixed point (the vertex) not in the plane of the circle. In other (modern) words, a line through P and a moving point on the circle sweeps out a cone.

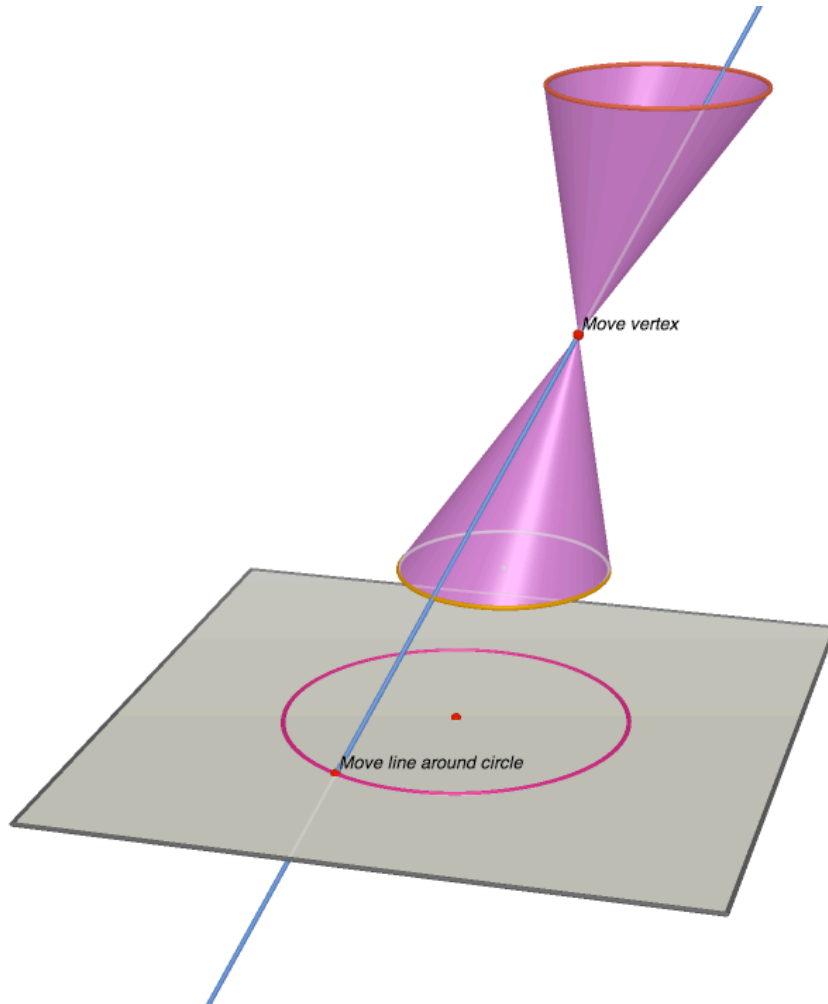


Figure 1. The cone generated by a circle and a point.

[Applet: [ConePtCircle.html](http://www.quadrivium.info/ConePtCircle.html)]

The circle is called the base, or the generating circle, of the cone. The point P is called the vertex of the cone.

Because a cone is defined by an entire line, it has two pieces, called nappes. This is important because some planes cut both the pieces (in a hyperbola). Also, the cone extends forever in two directions. It does not end at the circle like an ice cream cone.

If the line connecting the vertex of a cone to the center of the generating circle is at right angles to the plane of the circle, the cone is called a right cone. Otherwise it is an oblique, or skew, cone.

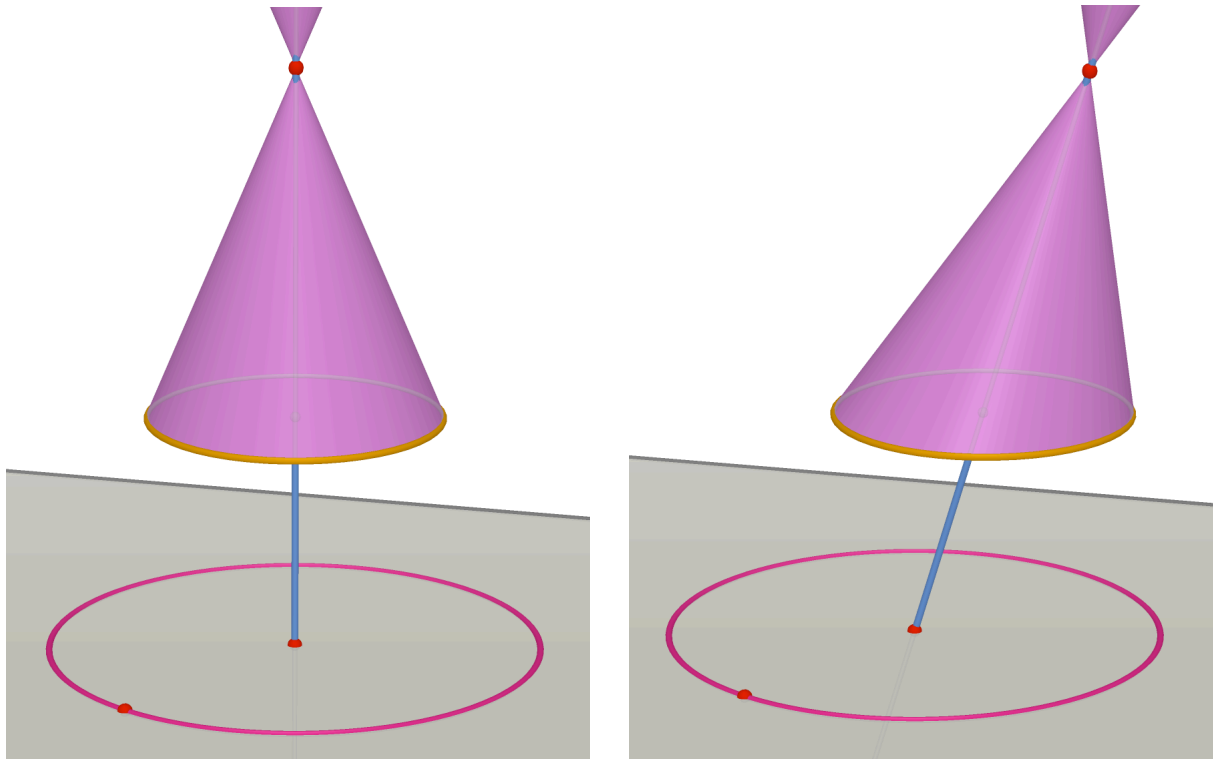


Figure 2. A right cone (at left) and an oblique cone (at right)

(Planes of) symmetry of a cone [Applet: ConeSym.html]

Any circular cone has two planes of reflection symmetry, which go through the vertex of the cone. One of these planes also goes through the center of the generating circle. If the cone is oblique, the other plane does not go through the center.

A view looking perpendicularly at the plane through the center can be considered as giving "the" side view of the cone, or its profile. In this view, you will see a non-isosceles triangle if the cone is oblique, or an isosceles triangle, if the cone is a right cone.

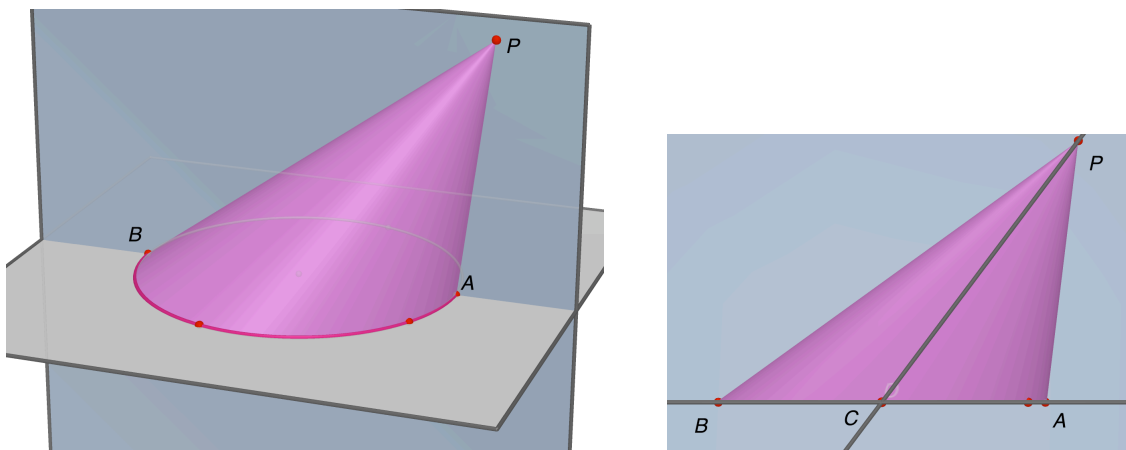


Figure 3. One plane (PAB) of symmetry (at left.) The "side view" (at right.)

In this picture, it doesn't look like plane PCD is a plane of symmetry because the cone is cut off (horizontally) at plane ABCD. However, keep in mind that the cone extends infinitely, top and bottom. The view from the "top", along line PO (where O is the center of the generating circle) suggests that this oblique cone on circle ABCD is a right cone on an ellipse! Note that circle ABCD appears to be an ellipse because it is viewed at a non-right angle.

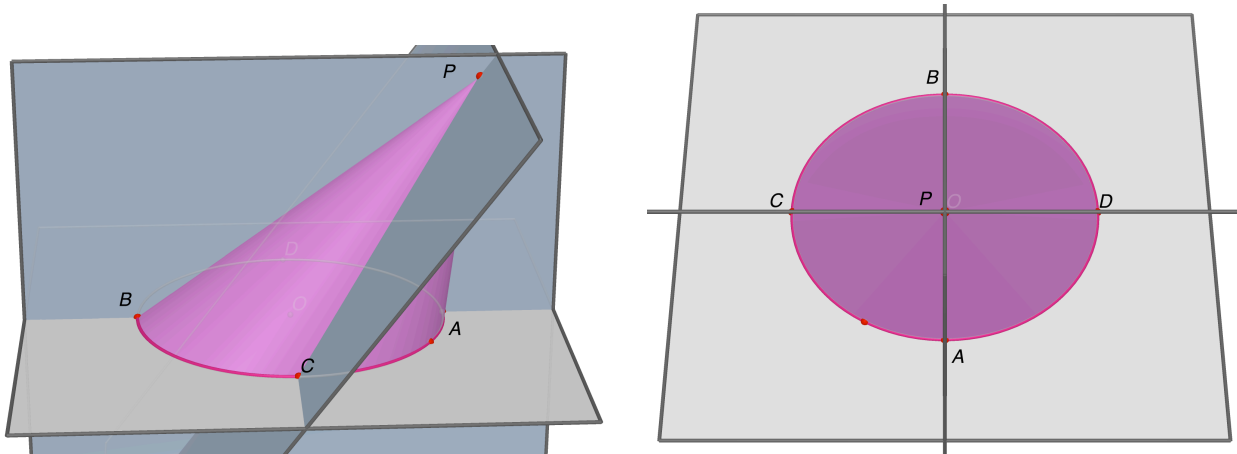


Figure 4. The second plane (PCD) of symmetry (at left.) The "top view" (at right.)

If the cone is a right cone, it has an infinite number of planes of reflection symmetry, all the planes through the vertex and the center of the circle. It also has rotation symmetry around the line through the vertex and the center, by any angle.

There are other symmetries that switch the two nappes of the cone.

Circular cross sections of a cone

In a right cone, all planes parallel to the fixed circle will have circular sections with the cone, and these are the only planes that make circular sections.

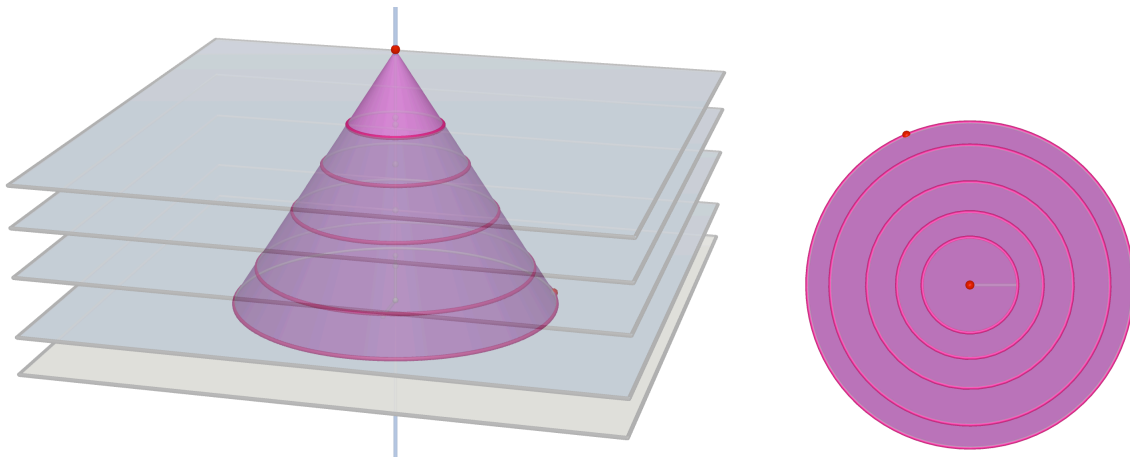


Figure 5. The circular sections to a right cone. At right, the top view.

However, in an oblique cone, there are exactly *two* families of parallel planes whose sections with the cone are circles. One of these families is parallel to the fixed generating circle and the other is called by Apollonius the subcontrary sections.

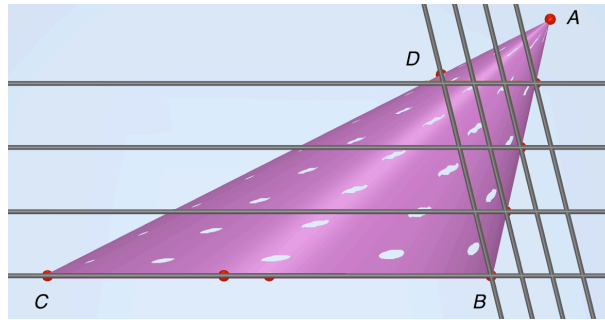


Figure 6. Side view of a cone with the two directions of circular sections.

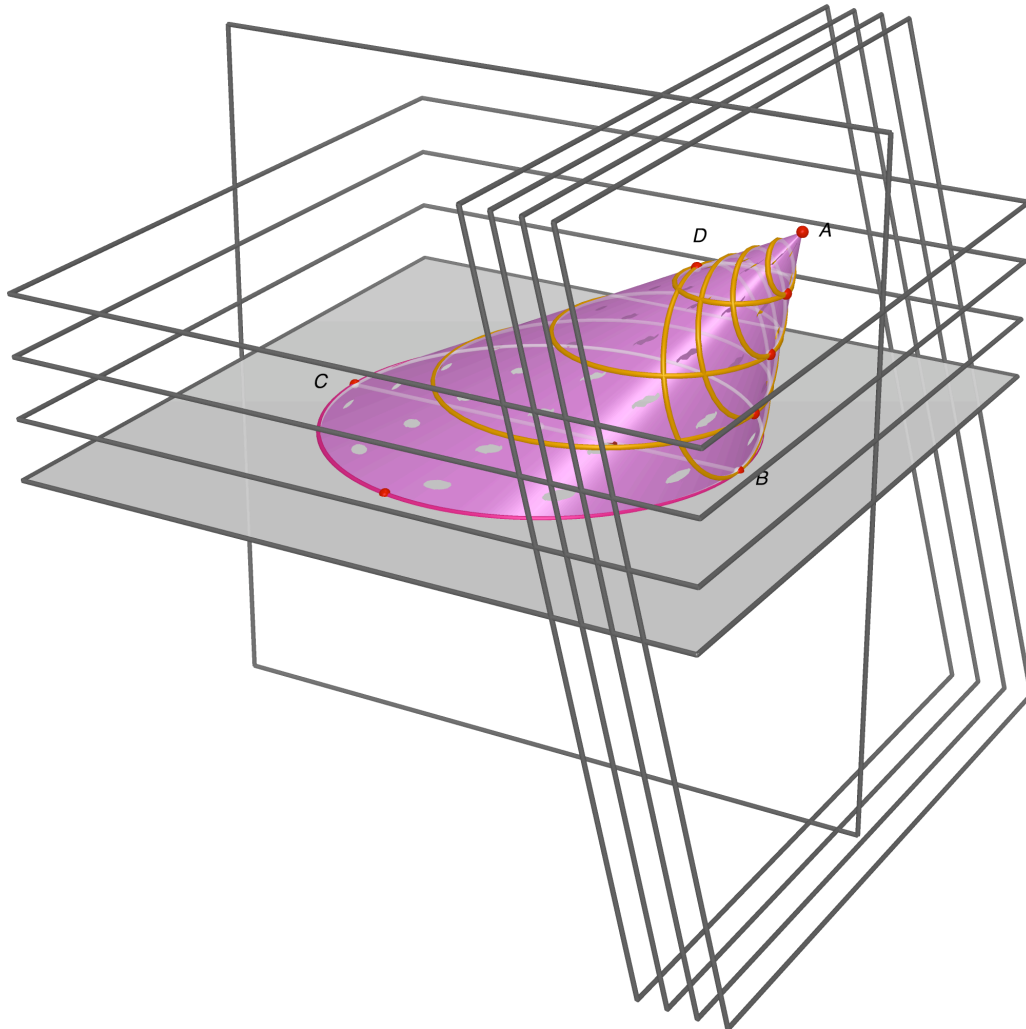


Figure 7. The same cone with two directions of circular sections.
[Applet: CircSecs2.html]

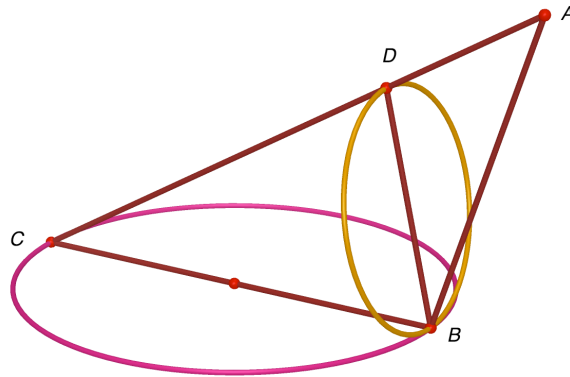


Figure 8. Triangles ABC and ADB are similar.

If one looks at the triangles formed by the diameters of the circular sections (both families) and the vertex of the cone, they are all similar. That is, if the triangles ABC and ADB in Figure 8 are similar, then all sections of the cone parallel to either one of these circles will be circles. This is Book 1, Proposition 5 in Apollonius. This is not an isolated oddity. Quite the contrary, it is the beginning of whole series of duality propositions about conics which are fundamental in many later approaches to curves and functions. As is often the case, similarity is at the heart of the issue.

Try this (1). What happens in Figure 8 if the cone is a right cone?

Tangent planes to a cone

Intuitively, a tangent plane to a surface just touches it at one point, and at that point, the surface and the plane have the same directions. (Here "directions" refer to tangent vectors.) Farther away, the surface may curve away, but the tangent plane is still straight.

Any tangent plane to a cone touches the cone at an entire line that goes through the vertex.

If a different plane cuts the cone at a point with a tangent plane, then the other plane intersects the cone in a curve, and it cuts the tangent plane at a line that is tangent to the curve.

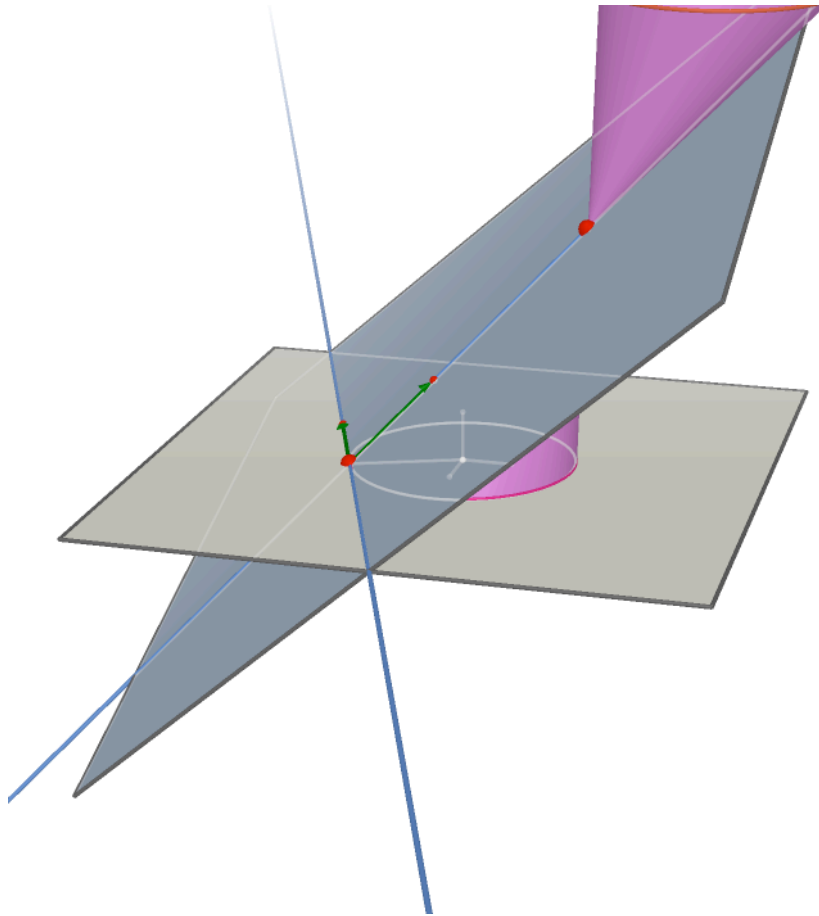


Figure 9. A tangent plane to a cone, showing the line through the vertex and the tangent line to a circle, and two tangent vectors.
 [Applet TangentPlane.html]

C. Apollonius's classification of parabolas as conic sections

Apollonius proved that when any cone is sliced by a plane which is parallel to one of its tangent planes (but not containing a generating line), then the resulting section is a parabola, in the sense that we know it as a curve: one of its coordinates is proportional to the square of the other. This argument is taken from Apollonius (Book 1, Proposition 11), but was known much earlier. This argument does not depend on the cone being a right cone. All that is needed is some set of parallel circular sections.

The proof in the parabola case

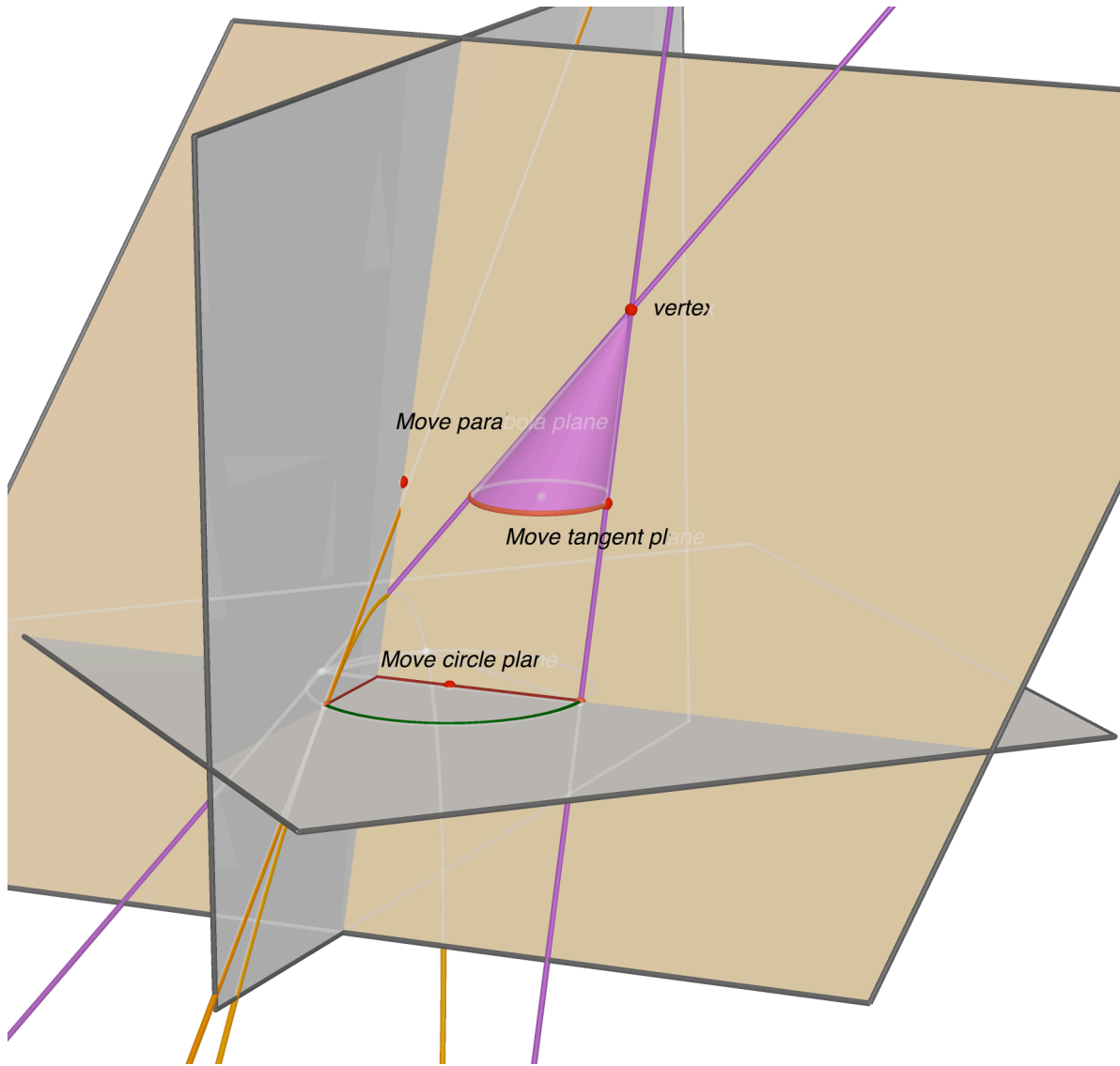


Figure 10. A cone cut by a plane, forming a parabola. Two other planes needed in the proof are also shown.
[Applet: ConeParabola.html]

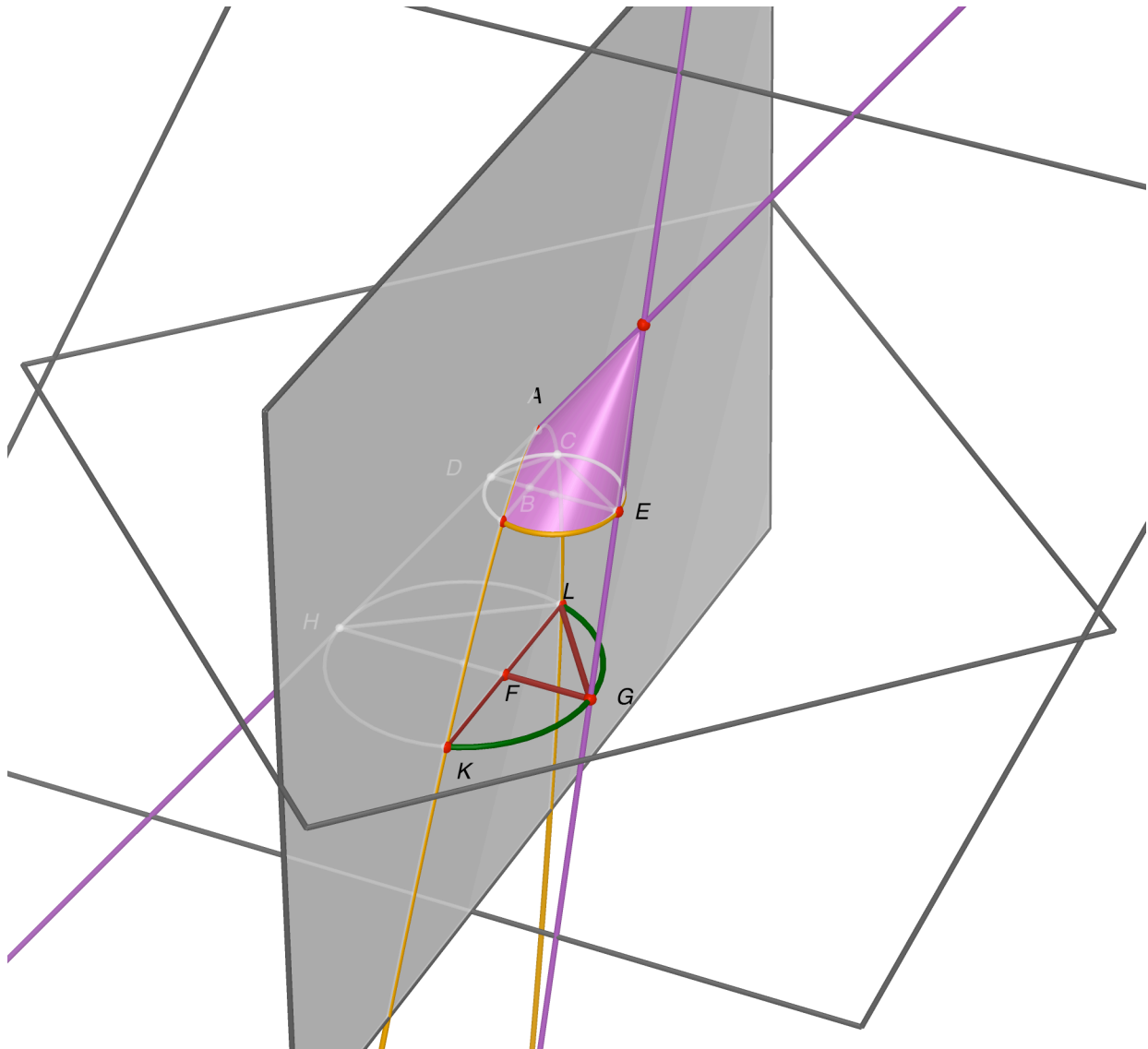


Figure 11. The same cone, points labeled, some planes transparent, for the proof.

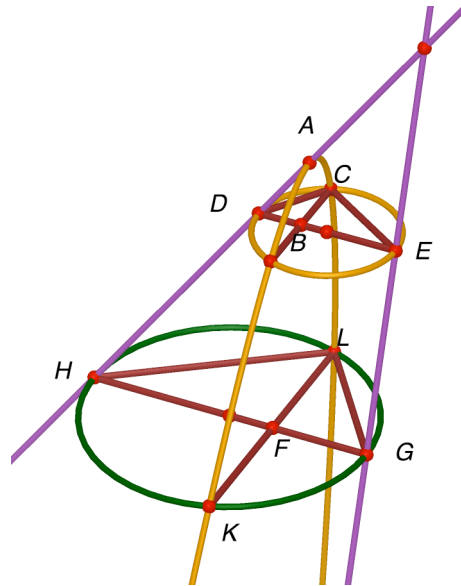


Figure 12. Lines, circles, and the parabola on the cone in Figure 11, for the proof.

In Figure 12 let the curve KAL be formed from slicing the cone with a plane parallel to the line \overline{EG} . Let GKH be a circular section and let D , C , and E lie on any parallel circular section (with C on the curve). Now LF is the geometric mean of HF and FG , i.e. $LF^2 = FH \times FG$. Likewise BC is the geometric mean of DB and BE , i.e. $BC^2 = BD \times BE$ (this is a fundamental property of the diameter of a circle and any chord perpendicular to it). Since the parabolic section is parallel to \overline{EG} , $BE = FG$. Since triangles ABD and AFH are similar to each other, we have the proportion $BD:BA = FH:FA$. If we think of sliding the parallel circular sections, and watching the two segments on the diameter, one of them is fixed ($BG = FE$), and the other (BD or FH) is proportional to the distance along the axis of the curve (BA or FA). Hence the distance along the axis of the parabola (BA or FA) is proportional to the square on the lateral distance out to the curve (BC or FL) by the geometric mean property of circles.

Introducing variables, we could let $AB = x$ and $BC = y$, and then say that there exists some constant of proportionality, p , such that $(p \cdot x) \cdot (BE) = y^2$. The constant, p , is the proportionality constant from the similar triangles just mentioned (i.e. $p = DB:AB$). It projects from the plane of the parabola (where x is measured) to the planes of the circular sections.

Of course Apollonius did not write an equation. Instead he showed how to construct a single line segment (called the latus rectum), and stated that the rectangle formed by this segment and \overline{AB} is always equal to the square on \overline{BC} . This latus rectum (whose length equals $p \cdot BE$) works for any parallel circular section along this parabola. The latus rectum is an actual line segment in his construction.

The view taken in this construction is that a parabola is generated by a changing set of geometric means where one of the segments is constant (in Figure 12 it is $BE = FG$ that is constant). In the cone, the circles which create these geometric means are not in the plane of the parabola, but the proportionality still holds there because of the similar triangles (i.e. $\triangle ABD \approx \triangle AFG$).

Outline of the proof for ellipses and hyperbolas

This view can be extended to the elliptic and hyperbolic cases by considering a linear proportional change in BE as we move down the cone. BE decreases in the elliptic case and increases in the hyperbolic case (see Figure 12). In the elliptic case the equation coming from the geometric means on the series of circular sections would be: $y^2 = (p \cdot x) \cdot (p' \cdot (k - x))$; where y , x , and p are as before; k is the entire finite length of the axis along which x is measured; and p' is a constant of proportionality that projects $k - x$ onto the circular sections in the plane containing AHF ($p = p'$ only in the case where the ellipse is formed as a section of a cylinder). This equation comes from the cone in the same way as before except that BE is replaced by $p' \cdot (k - x)$. This equation can be rewritten as:

$$\frac{y^2}{p \cdot p' \cdot \left(\frac{k}{2}\right)^2} + \frac{\left(x - \frac{k}{2}\right)^2}{\left(\frac{k}{2}\right)^2} = 1,$$

which is the standard form given in modern textbooks, but with a translation because x is measured from a vertex at one end of the axis instead of from the center. For the hyperbola, simply replace $p' \cdot (k - x)$ with $p' \cdot (k + x)$ to indicate the increasing lengths of BE . k in this case is the distance between the vertex on the hyperbola and the vertex of the opposite branch. Again it should be noted that Apollonius did not write equations, but made these statements in ratio form with geometrically constructed adjustments to the latus recta. One should also note that Apollonius gave the converse of these arguments, i.e. that any curve with these geometric mean properties (equations) must be a conic section, because the geometric means can always be constructed from a series of circles whose radii are changing arithmetically (hence a cone).

D. Arabic constructions of conic sections

A tenth century Arabic mathematician, Ibn Sina, wrote several commentaries on Apollonius in which he showed how to draw conics using ruler and compass. He simply flattened the picture, and put the series of circles in the same plane as the curve. His method was still the same. He built, for example, a parabola using a series of geometric means where one of the segments is held constant. Looking at Figure 13, one sees a series of tangent circles all passing through the point S . A fixed vertical line at A constructs the geometric means between SA (constant) and a series of segments \overline{AT} , \overline{AU} , \overline{AV} . These segments are then plotted against the series of geometric means AX , AY , AZ , to give a set of points A , B , C , D , all of which lie along a parabola. In this picture the horizontal distances of the points (B , C , and D) from A is proportional to the square on their vertical distances. The latus rectum is SA .

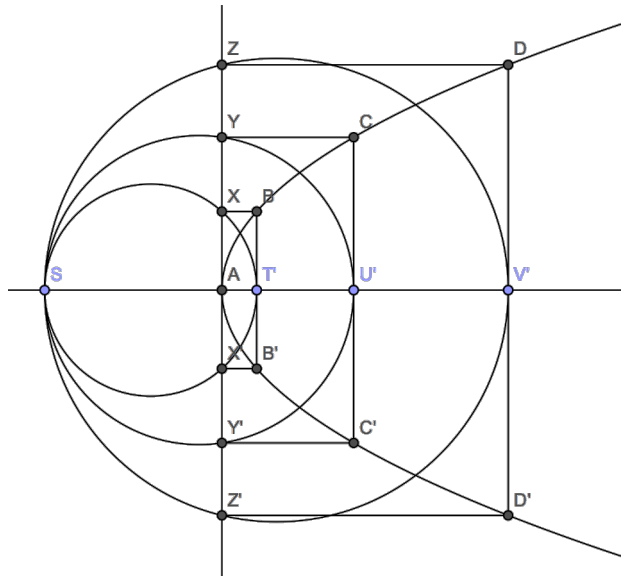


Figure 13. Ibn Sina's parabola construction
 [Applet: IbnSinaParabola.html]

This and other Arabic methods of drawing conics were static in the sense that they allowed one to plot as many points on the curves as desired, but they did not draw the curve in a continuous way. This particular construction of Ibn Sina, however, can be easily adapted as a design for a dynamic mechanism which will draw parabolas in a smooth continuous manner. Imagine a nail driven in at S , and a carpenter's square with the vertex of its right angle riding on the line \overline{AX} and one arm resting against S . As we move the vertex X along the vertical line the other arm of the square will be intersecting the line SA at points T such that AX is always the geometric mean of SA and AT . Using a second carpenter's square XBT we can continuously locate the points on the parabola. Figure 14 shows two positions of an animation of such a device. The point X is being moved along the vertical line to drive the animation while the locus of the point B draws the parabola.

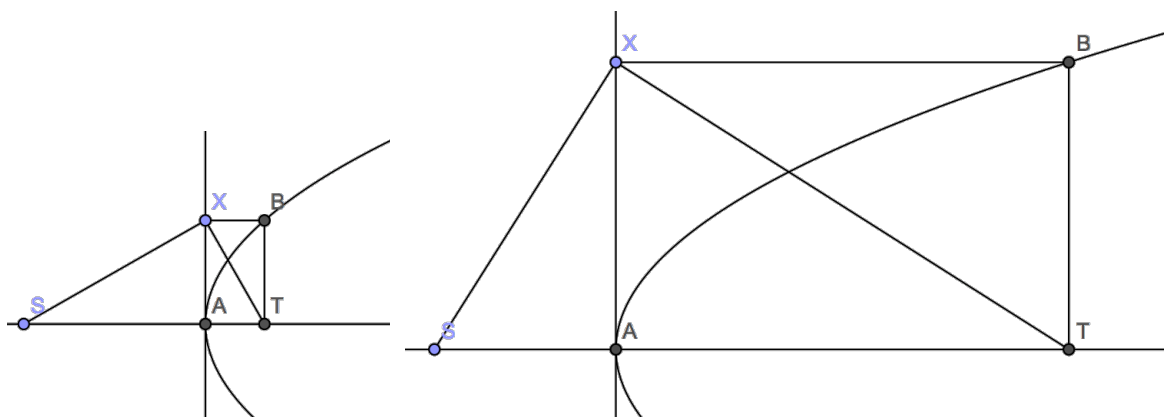


Figure 14. A modern dynamic adaptation of Ibn Sina's method.
 [Applet: 2RtTriParabola.html]

This dynamic adaptation of Ibn Sina's construction was not taken from any historical source. It is similar in nature, however, to many of the devices from the seventeenth century described in the next section. These examples allow the reader to see a transition from classical geometry to a dynamic situation where actions generate curves. An analysis of such actions provides important groundwork for understanding the evolution of modern notions of analytic geometry and functions.

Arabic interest in drawing conic sections was spurred by their development of methods for solving cubic equations by intersecting two conic sections (e.g. by intersecting a parabola with a hyperbola). In the work of Omar Khayyam, a general geometric method for the solution of cubics is given. These methods had a profound impact on later Western mathematics. For example, one of the primary goals of Descartes' *Geometry* was to extend these Arabic methods by finding new ways to draw curves whose intersections would yield solutions to equations of arbitrary degree. One should keep in mind that equations to be solved were frequently stated first as geometry problems (e.g. doubling a cube, trisecting an angle, or finding the parameters of curves that meets certain tangent requirements). This was true for both Arabic and European mathematicians up to 1650.

The new views that transformed European mathematics in the seventeenth century were not just a move from geometry to algebra, but perhaps more importantly a move from a static to a dynamic geometry that involved time and motion.

E. Conjugate diameters and equations for conics

In the opening definitions of *The Conics* the following definition appears:

4. Of any curved line which is in one plane I call that straight line the *diameter* which, drawn from the curved line, bisects all straight lines drawn to this curved line parallel to some straight line; and I call the end of that straight line (the diameter) situated on the curved line the *vertex* of the curved line, and I say that each of these parallels is drawn *ordinatewise* to the diameter

Mathematicians in the 1600s, before the concept of a coordinate system emerged, used the terms *abscissa* and *ordinate* as proto-coordinates for a curve. Given a curve C and axes (lines) L and M intersecting at O , each point P on the curve determines two line segments: PQ , along a parallel to M , and OQ , along L . Points P_i along C , in order, give an ordered sequence of segments P_iQ_i (the ordinates), and corresponding segments OQ_i cut off along L . (The word *abscissa* means literally "cut off" in Latin.)

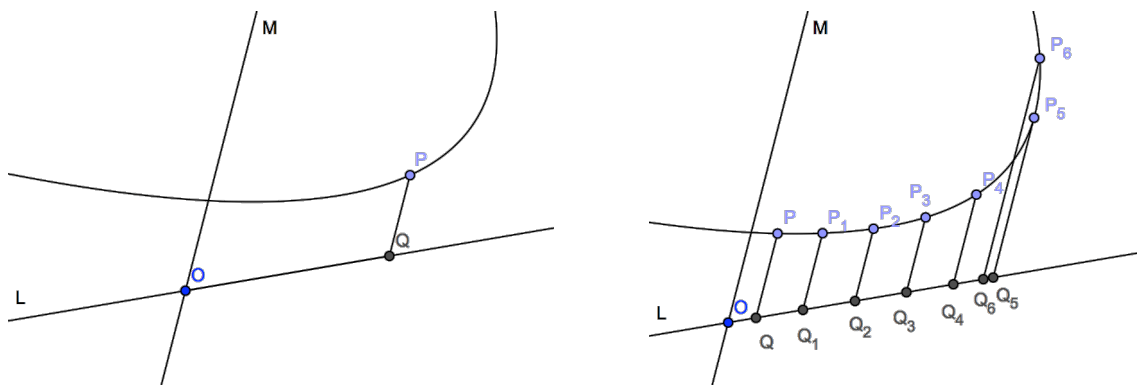


Figure 15. Axes L and M , abscissas OQ , ordinates QP .

It turns out that, in the case of circles, ellipses, and hyperbolas, the diameters are exactly the lines that pass through the center. In the case of the circle, the ordinates are always perpendicular to the diameter, but in the case of other conics this is not always so (see). The ordinatewise direction is the set of chords which are all parallel to the tangent line at the vertex of the diameter. It was surprising to me, at first, that such a family of bisected chords existed except along an axis of symmetry. This is never mentioned in modern discussions of conic sections, except in certain projective treatments. Apollonius defines an "axis" as a special case of a diameter where the family of bisected chords are perpendicular to the diameter.

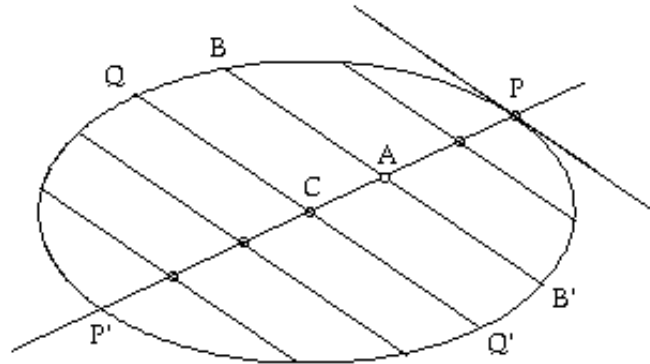


Figure 16.

[Applet: EllConjDiam.html]

Figure 16 shows an ellipse with center C . Choosing an arbitrary point P on the ellipse, \overline{PC} is a diameter according to the Definition 4, because all of the chords parallel to the tangent at P will be bisected by \overline{PC} . Another property of conics demonstrated in Apollonius is that the chord $\overline{QQ'}$ which passes through the center C has as its ordinates (i.e. bisected chords) all chords parallel to \overline{PC} . Hence the tangent at Q is also parallel to \overline{PC} . Apollonius called $\overline{QQ'}$ the conjugate diameter to $\overline{PP'}$. This is an example of a fundamental duality property of all conics.

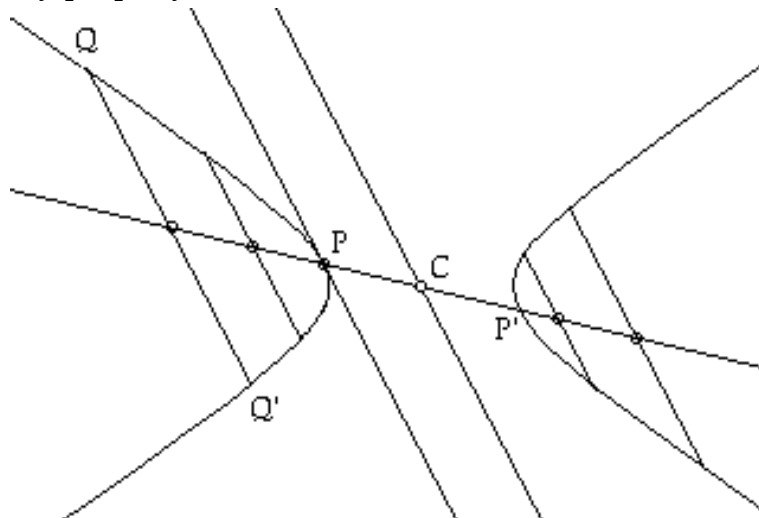


Figure 17.

Let us look next at these bisected chords and conjugate diameters on the hyperbola (see Figure 17). It remains true that any line through the center, C , will bisect all chords (e.g. $\overline{QQ'}$) parallel to the tangent at the vertex P . There is a line through center that is parallel to the ordinates with respect to \overline{PC} , but which does not intersect the curve. It is still, however, a conjugate diameter in the sense that any chord parallel to \overline{PC} will be bisected by this line through C . Such chords (e.g. $\overline{PP'}$) go between the two opposite branches of the hyperbola instead of being contained within one branch. Although this "conjugate diameter" does not intersect the curve, Apollonius gives a finite length centered at C so that it has the same ratio properties as in the elliptic case. This is a generalized version of what is usually done in classrooms when a specific rectangle is constructed in the middle of a hyperbola having the asymptotes as its diagonals.

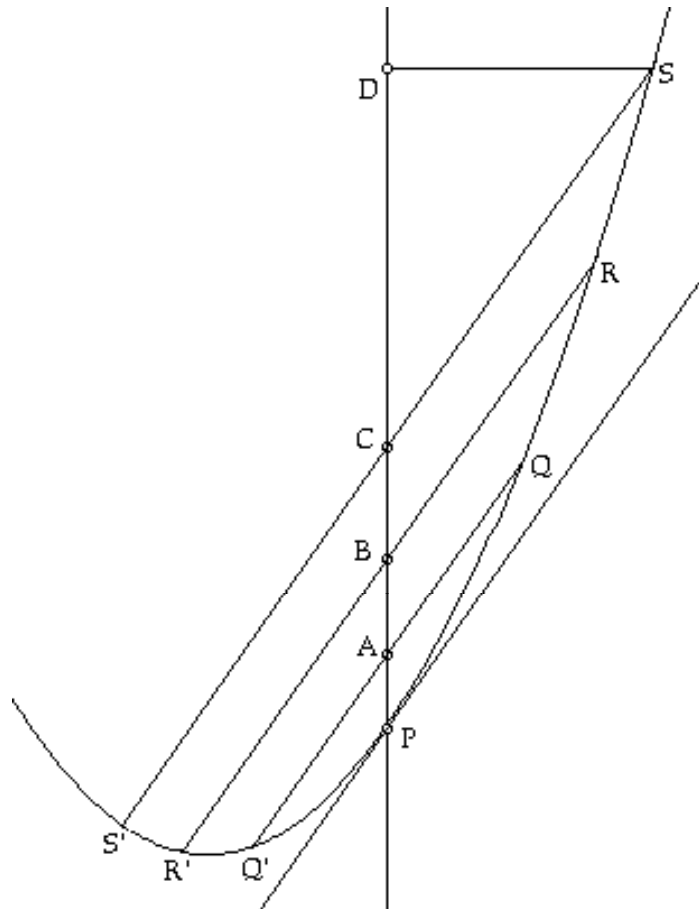


Figure 18.

[Applet: ParabConjDiam.html]

Turning to the parabola adds more justification to Apollonius' definition of a diameter. A parabola has no center (it is at infinity) but it does have many diameters. Without projective methods, these diameters cannot be defined as lines through the center. They turn out to be all lines which are parallel to the axis of symmetry (see Figure 18). In this case none of the diameters intersect each other (unless one includes the point at infinity), so there are no pairs of conjugate diameters, but any vertical line

in Figure 18 (e.g. \overline{PC}) will bisect all chords parallel to the tangent at its vertex P (e.g. $AQ = AQ'$, $BR = BR'$, and $CS = CS'$).

Apollonius demonstrated ratio properties for each conic section, that, if translated into algebra, and applied to the axes (in the Apollonian sense), would yield our modern standard equations. He demonstrated, however, that these ratio properties are true for any of his diameters. In modern algebra this implies that the form of the equations of conic sections remains unchanged if we coordinatize the curve along different diameters, provided we always use the ordinate direction appropriate to the diameter (i.e. a non-perpendicular coordinate system using the direction of the bisected chords). Only the constants (e.g. the latus rectum in the parabolic case) will have to be adjusted. When we use our standard equations of conics (coordinatized perpendicularly along the axes of symmetry), we can choose either the positive or negative square root to find pairs of symmetrical points. This Apollonian bisection property shows that this can also be done using any diameter, even if it is not an axis of symmetry.

For example, in Figure 18, if we use the diameter at P , then for any point C , we will have that $CS^2 (= CS'^2)$ will be proportional to PC . The right triangle CSD will give us the appropriate new proportionality constant, relative to the axis of symmetry, by taking the ratio of the squares of CS to DS . For example let us assume that the parabola in Figure 18 has the equation $1 \cdot y = x^2$ with respect to the (axial) perpendicular coordinates with the origin at the vertex. Suppose we then construct the diameter (vertical line) through the point $P = (1,1)$ and use parallel chords all having a slope of $+2$ as our ordinates. Supposing that $P = (1,1)$ in Figure 18, that is to say, let the coordinates of the point Q on the parabola, be given in the new system as $x' = AQ$ and $y' = PA$; or of the point S , as $x' = CS$, and $y' = PC$. The equation of the parabola in these coordinates will be $5 \cdot y' = x'^2$, because the ratio $CS^2:DS^2 = 5:1$ (coming from a right triangle with legs in a 2:1 ratio). For example, the points $(2,4)$ in the axis system becomes $(\sqrt{5}, 1)$ with respect to the diameter through P . The point $(-\sqrt{5}, 1)$ is on the curve at the other end of the bisected chord. Putting the constant (i.e. 5) with y preserves the dimensional integrity of the statement (i.e. it says that a rectangle equals a square). Apollonius would say that the latus rectum with respect to the new diameter is five times the latus rectum with respect to the axis. Such a statement avoids choosing a unit.

To demonstrate the statements in the previous paragraph, look back at Figure 12. If the cone is oblique such that the plane containing A , H , and G is not perpendicular to the plane of the base circle $KHLG$, then AF will not be the axis of the parabola but it will still be a diameter (Apollonius, Book 1, Prop. 7). BC and FL will still be ordinates with respect to AF , but they will not be perpendicular to AF (they will still appear as in Figure 18). The previous discussion of Figure 12 remains valid since it depended only on having a series of parallel circular sections which produce a series a geometric means. AB is still proportional to BC^2 . The form of the equations remains the same. The same is true for ellipses and hyperbolas. Ratio and similarity are more flexible and general than they often appear in our usual curriculum.

F. Exercises

Try this (1). Prove all the results in section B, in Euclidean style. What axioms/postulates should you use?

For the next problems, use modern coordinate geometry.

Try this (2). Let C be a curve in the plane $z=1$ defined by a polynomial (in x and y .) For example, the curve defined by the equations $x^2 + y^2 = 1$ and $z = 1$ is a unit circle in the plane $z=1$, and it is defined by a polynomial of degree 2. Multiply each term by a power of z so that each term has degree 2. (It may be the 0th power. This is called *homogenizing* the curve. It produces a homogeneous equation, in which all terms have the same degree.) In the circle example, you get $x^2 + y^2 = z^2$. A cone on any plane curve is defined to be the set of points on all lines connecting points on the curve with a vertex not in the plane. Explain why the homogenized equation for C is the cone on C with vertex $(0,0,0)$.

Try this (3). Find an equation for an oblique cone on a circle in which the vertex is directly above (or below) a point on the circle.

Try this (4). (Requires a serious course in linear algebra, including eigenvalues and diagonalization of quadratic forms.) Use the cone in [Try this (3)]. Prove that it is a right cone on an ellipse, and give the ellipse and its plane explicitly. (Once you have set up the problem, a computer algebra system, such as Maple or Mathematica, is highly recommended. The calculations are pretty horrendous.)

G. References

The books from Green Lion press are very careful translations, with modern commentary. The Heath translation is from Victorian England, and is more of a rewrite than a translation.

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Apollonius of Perga, & Donahue, W. H. (1998). *Conics (Revised.)*. Green Lion Press.

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Newton's *Principia Mathematica* is notoriously difficult to read because it has Apollonius as a prerequisite. The Densmore edition attempts to fill in this background.

Densmore, N. B. D. (2003). *Newton's Principia: The Central Argument* (3rd ed.). Green Lion Press.

Newton, I. (1975). *The Mathematical Principles of Natural Philosophy and His System of the World*. Berkeley: University of California Press.